An Introduction to Partial Differential Equations in the Undergraduate Curriculum

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LECTURE 12
Heat Transfer in the Ball

12.1. Outline of Lecture

- The problem
- The problem for radial temperatures
- Solution by separation of variables
- Interchanging infinite sums and limits

12.2. The Problem

By the ball of radius a we mean the set B in $\mathbf{R}^3$ defined by

$$B = \left\{ (x,y,z) \in \mathbf{R}^3 \middle| x^2 + y^2 + z^2 \le a^2 \right\}.$$

We are interested in studying how the temperature $u(x,y,z,t)$ varies from point to point and with the time t. We already know that the temperature must satisfy the heat equation

$$u_t = k\Delta u = k\left[u_{xx} + u_{yy} + u_{zz}\right], \tag{12.1}$$

where $k > 0$ is a constant called the thermal diffusivity.

In general we are required to supply the initial temperature distribution,

$$u(x,y,z,0) = u_0(x,y,z) \quad \text{for } (x,y,z) \in B, \tag{12.2}$$

and the temperature on the boundary,

$$u(x,y,z,t) = f(x,y,z) \quad \text{for } (x,y,z) \in \partial B \text{ and } t > 0. \tag{12.3}$$

Of course, the boundary of the ball is the sphere

$$S = \partial B = \left\{ (x, y, z) \in \mathbf{R}^3 \middle| x^2 + y^2 + z^2 = a^2 \right\}.$$

One example of this phenomenon involves deciding how long it takes to cook a turkey (approximately spherical) that has been defrosted, allowed to reach room temperature, and then is placed into an oven where a constant ambient temperature is maintained. A second problem of the same type arises when the turkey is taken out of the oven and allowed to sit with its surface exposed to room temperature while its interior temperature distribution is what has been reached in the oven. In this case the center of the turkey continues to heat up for some time while the outer portion is cooling.

The problem in equations (12.1), (12.2), and (12.3) is called the *Dirichlet problem* for the heat equation in the ball. The solution is typically achieved in two stages. First we find the *steady-state temperature* u_s. This is a temperature which is independent of time and has the same boundary conditions as u. Since u_s satisfies $\partial u_s/\partial t = 0$ and also satisfies the heat equation (12.1)), we must have $\Delta u_s = 0$. Therefore u_s solves the problem

$$\begin{aligned} \Delta u_s(x, y, z) &= 0 \quad \text{for } (x, y, z) \in B, \text{ and} \\ u_s(x, y, z) &= f(x, y, z) \quad \text{for } (x, y, z) \in \partial B. \end{aligned} \tag{12.4}$$

The problem in (12.4) is called the Dirichlet problem for the Laplacian, Δ.

The second stage in the solution is to find the difference $v(x, y, z, t) = u(x, y, z, t) - u_s(x, y, z)$, which might be referred to as the *transient* temperature. Putting together the information in (12.1), (12.2), (12.3), and (12.4), we see that v must solve the problem

$$\begin{aligned} &v_t(x, y, z, t) = k\Delta v(x, y, z, t) \quad \text{for } (x, y, z) \in B \text{ and } t > 0, \\ &v(x, y, z, 0) = u_0(x, y, z) - u_s(x, y, z) \quad \text{for } (x, y, z) \in B, \text{ and} \\ &v(x, y, z, t) = 0 \quad \text{for } (x, y, z) \in \partial B \text{ and } t > 0, \end{aligned} \tag{12.5}$$

Comparing (12.5) to the problem described in (12.1), (12.2), and (12.3), we see that v is a solution to the Dirichlet problem for the heat equation with homogeneous boundary conditions. This will enable us to use separation of variables in our solution in Section 12.4.

Having solved the problems in (12.4) and (12.5), the solution of the Dirichlet problem for the heat equation is $u(x, y, z, t) = u_s(x, y, z) + v(x, y, z, t)$.

12.3. The Problem for Radial Temperatures

The problems in (12.4) and (12.5) are rather daunting in the generality we have presented. We will limit ourselves to the case when the initial temperature u_0 and the boundary temperature f are both constants.

Let's turn first to the problem in (12.4) of finding the steady-state temperature, but now with a constant temperature f given on the boundary. Using either physical intuition or mathematical insight we are led to the suggestion that the steady-state temperature will also be a constant. We can verify directly that the constant function $u_\mathrm{s}(x, y, z) = f$ is a solution to (12.4).

Next, in (12.5) we are looking for a temperature v which is equal to 0 everywhere on the boundary, and is initially equal to the constant $v_0 = u_0 - f$. Let's consider this solution along a radius of the ball. Since the initial and boundary conditions do not distinguish one radius from another, we are led to expect that the temperature distribution v will be the same along any radius. Therefore, in the spatial coordinates, v will depend only on the distance from the center of the ball, which is $r = \sqrt{x^2 + y^2 + z^2}$. Let's look for a solution to (12.5) of the from $v(r, t)$.

We need to compute Δv for a function of this form. If g is any function that depends only on r, so that $g = g(r)$, then by the chain rule

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial r} \cdot \frac{\partial r}{\partial x} = g_r r_x. \tag{12.6}$$

We need to compute r_x. That is most easily done by differentiating both sides of

$$r^2 = x^2 + y^2 + z^2 \tag{12.7}$$

to get

$$2 r r_x = 2x \quad \text{or} \quad r_x = \frac{x}{r}.$$

Similarly,

$$r_y = \frac{y}{r} \quad \text{and} \quad r_z = \frac{z}{r}.$$

Therefore from (12.6), if g depends only on r,

$$\frac{\partial g}{\partial x} = \frac{x}{r} \frac{\partial g}{\partial r}. \tag{12.8}$$

In particular, $v_x = (x/r) v_r$. Using the product formula and (12.8) , we compute that the second derivative is

$$v_{xx} = \frac{\partial v_x}{\partial x} = \frac{\partial}{\partial x} \left(\frac{x}{r} \cdot v_r \right)$$

$$
\begin{aligned}
&= \frac{\partial}{\partial x}\left(\frac{x}{r}\right)\cdot v_r + \frac{x}{r}\cdot\frac{\partial v_r}{\partial x}\\
&= \frac{r - x(x/r)}{r^2}\cdot v_r + \frac{x}{r}\cdot\frac{x}{r}\frac{\partial v_r}{\partial r}\\
&= \frac{r^2 - x^2}{r^3}v_r + \frac{x^2}{r^2}v_{rr}.
\end{aligned}
$$

The second derivatives with respect to y and z have similar formulas. Adding them together using (12.7) we get

$$\Delta v = v_{xx} + v_{yy} + v_{zz} = \frac{2}{r}v_r + v_{rr}. \tag{12.9}$$

Thus, to solve (12.5) we are looking for a function $v(r,t)$ which satisfies

(12.10)

$$
\begin{aligned}
&v_t(r,t) = k\left[v_{rr}(r,t) + \frac{2}{r}v_r(r,t)\right] \quad \text{for } 0 \le r < a \text{ and } t > 0,\\
&v(r,0) = v_0 \quad \text{for } 0 \le r < a,\\
&v(a,t) = 0 \quad \text{for } t > 0.
\end{aligned}
$$

12.4. Solution by Separation of Variables

In view of past experience, it is natural to look for product functions $v(r,t) = R(r)T(t)$ which satisfy the differential equation and the boundary condition in (12.10), or

(12.11)

$$
\begin{aligned}
&v_t(r,t) = k\left[v_{rr}(r,t) + \frac{2}{r}v_r(r,t)\right] \quad \text{for } 0 \le r < a \text{ and } t > 0,\\
&v(a,t) = 0 \quad \text{for } t > 0.
\end{aligned}
$$

Separating variables in the usual way, we find that T must satisfy

$$T' = -\lambda k T, \tag{12.12}$$

while R must satisfy

$$-\left[R'' + \frac{2}{r}R'\right] = \lambda R \quad \text{and } R(a) = 0, \tag{12.13}$$

where λ is a constant.[1] The solution to (12.12) is

$$T(t) = e^{-\lambda k t}. \tag{12.14}$$

[1]The functions T and R are each functions of one variable although the variable is different for each. We use the prime notation to indicate the derivative with respect to the one variable. Thus $T' = dT/dt$ and $R' = dR/dr$.

To find the solution to (12.13) we must work a little harder, but the solution is surprisingly easy.

First of all we must make the differential equation in (12.13) look like a Sturm-Liouville equation. This means we want to find a function $p(r)$ so that

$$p\left[R'' + \frac{2}{r}R'\right] = [pR']' = pR'' + p'R'.$$

This requires $p' = 2p/r$, so $p(r) = r^2$ will work. Multiplying the differential equation in (4.3) by r^2, it becomes

$$-\left[r^2R'' + 2rR'\right] = -\left[r^2R'\right]' = \lambda r^2 R. \tag{12.15}$$

Equation (12.15) now has the form of a Sturm-Liouville equation with weight function r^2. Notice that the coefficient $p(r) = r^2$ vanishes at $r = 0$, so this is a singular Sturm-Liouville equation.

Equation (12.13)) gives only the one boundary condition, $R(a) = 0$. However, there is a hidden condition that we see when we realize that the function $v(r,t) = T(t)R(r)$ is a temperature and has a finite value at $r = 0$. This means that $R(0)$ is also finite. Hence the complete Sturm-Liouville problem for R is

$$\begin{aligned} -\left[r^2R'' + 2rR'\right] = -\left[r^2R'\right]' &= \lambda r^2 R, \\ R(0) &\text{ is finite}, \\ R(a) &= 0. \end{aligned} \tag{12.16}$$

We can simplify the differential equation in (12.16) considerably by making the substitution[2]

$$S = rR. \tag{12.17}$$

Then $S' = rR' + R$ and $S'' = rR'' + 2R'$. Making this substitution into the differential equation, we get $-rS'' = \lambda rS$, or $-S'' = -\lambda S$. From (12.16) and (12.17), the boundary conditions satisfied by S are $S(0) = 0 \cdot R(0) = 0$ and $S(a) = a \cdot R(a) = 0$. To sum up, the function $S(r) = rR(r)$ must satisfy

$$-S'' = \lambda S, \quad \text{with} \quad S(0) = S(a) = 0.$$

[2]While this substitution comes out of the blue here, mathematicians have discovered that if we have a Sturm-Liouville equation $(pu')' + qu = \lambda su$ on an interval of the form $(0, b)$, which is singular at the initial end point $x = 0$, and where the coefficient p can be factored as $p(x) = x^{2\alpha}P(x)$, where $P(0) \neq 0$, the substitution $w(x) = x^{\alpha}u(x)$ will sometimes change the problem to a Sturm-Liouville problem that is regular at $x = 0$.

We have seen this Sturm-Liouville problem several times. The solutions are

$$\lambda_n = \frac{n^2\pi^2}{a^2} \quad \text{and} \quad S_n(r) = \sin\frac{n\pi r}{a}, \quad \text{for } n = 1, 2, 3, \ldots.$$

Since $R = S/r$, the solutions to the Sturm-Liouville problem in (12.16) are

$$\lambda_n = \frac{n^2\pi^2}{a^2} \quad \text{and} \quad R_n(r) = \frac{1}{r}\sin\frac{n\pi r}{a}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{12.18}$$

Notice that the apparent singularity at $r = 0$ is not really there. We will set

$$R_n(0) = \lim_{r\to 0} R_n(r) = \lim_{r\to 0} \frac{\sin(n\pi r/a)}{r} = \frac{n\pi}{a}. \tag{12.19}$$

Let's say a few words about orthogonality that are not directly related to our train of thought. We have proved directly that

$$\int_0^a S_i(r)S_j(r)\,dr = \int_0^a \sin\frac{i\pi r}{a}\,\sin\frac{j\pi r}{a}\,dr = 0 \quad \text{if } i \neq j.$$

From Sturm-Liouville theory, we know that the eigenfunctions R_j of the problem in (12.16) are orthogonal with respect to the weight r^2. Thus

$$\int_0^a R_i(r)R_j(r)\,r^2\,dr = 0 \quad \text{if } i \neq j.$$

Since $rR_j = S_j$, this is in exact agreement with the orthogonality relation for S_j. Finally, Let's look at the orthogonality of the functions $R_j(r)$ on the ball B, where $r = \sqrt{x^2+y^2+z^2}$. Using polar coordinates to do the integration and integrating out the angles ϕ and θ, we have

$$\begin{aligned}\iiint_B R_i(r)R_j(r)\,dxdydz &= \int_0^a\int_0^{2\pi}\int_0^{\pi} R_i(r)R_j(r)\,r^2\sin\phi\,d\phi d\theta dr\\ &= 4\pi\int_0^a R_i(r)R_j(r)\,r^2\,dr\\ &= 0 \quad \text{if } i \neq j.\end{aligned}$$

The last line follows from the orthogonality relationship for the R_j with respect to the weight r^2. It is the orthogonality on the ball B which is really important here. Unfortunately, we do not have time to explore this.

Let's return to the solution of the heat equation. From (12.14) and (12.18) we see that the product solutions of (12.11) are

$$v_n(r,t) = e^{-kn^2\pi^2t/a^2}\frac{\sin(n\pi r/a)}{r}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{12.20}$$

By superposition a convergent infinite series of the form

$$v(r,t) = \sum_{n=1}^{\infty} A_n v_n(r,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/a^2} \frac{\sin(n\pi r/a)}{r} \tag{12.21}$$

is also a solution to (12.11).

To complete the solution to (12.8), we must choose the coefficients A_n so that $v(r,0) = v_0$. Multiplying (4.11) by r and evaluating at $t = 0$, this becomes

$$v_0 r = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi r}{a} \quad \text{for } 0 < r < a.$$

This is the Fourier sine series for the function $v_0 r$. The coefficients are $A_n = (-1)^{n+1} \cdot 2av_0/\pi n$, so

$$v_0 = 2v_0 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n\pi r/a)}{n\pi r/a}.$$

Notice for later reference that with $v_0 = 1/2$ this becomes

$$\frac{1}{2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n\pi r/a)}{n\pi r/a} \quad \text{for } 0 < r < a \quad . \tag{12.22}$$

Inserting the coefficients A_n into (12.21), we see that the solution to (12.5) is

$$v(r,t) = 2v_0 \sum_{n=1}^{\infty} (-1)^{n+1} e^{-kn^2\pi^2 t/a^2} \frac{\sin(n\pi r/a)}{n\pi r/a}. \tag{12.23}$$

Recall that in our two stage approach, the solution to the original Dirichlet problem for the heat equation is the sum of the steady-state solution and the transient solution. Thus

$$u(r,t) = u_{\mathrm{s}}(r,t) + v(r,t) = f + 2(u_0 - f) \sum_{n=1}^{\infty} (-1)^{n+1} e^{-kn^2\pi^2 t/a^2} \frac{\sin(n\pi r/a)}{n\pi r/a}. \tag{12.24}$$

Example. Let's look at a special case. Assume that we are cooking a turkey, roughly the shape of a sphere of radius 1 foot. The turkey has been defrosted and is uniformly at room temperature of 75°. We put it into the oven at a temperature of 350°. We want to cook it until the center has a temperature of 150°. With time measured in hours, the thermal diffusivity in the proper units is $k = 0.02$. How long should we cook the turkey?

We have all of the information we need. The parameters are $f = 325$, $u_0 = 75$, $a = 1$, and $k = 0.02$. The solution is given in equation (12.24). Using Matlab, we sum the first 200 terms of the series in (12.24), and plot the results versus r at time intervals of 1 hour. The result is shown in Figure 1.

A more efficient way to find the time when the center of the turkey is at 150° is to plot that temperature versus time. Setting $r = 0$ in (12.24) and using (12.19) we see that

(12.25)
$$\begin{aligned} u(0,t) &= \lim_{r\to 0} u(r,t) \\ &= f + 2(u_0 - f)\sum_{n=1}^{\infty}(-1)^{n+1}e^{-kn^2\pi^2 t/a^2}\lim_{r\to 0}\frac{\sin(n\pi r/a)}{n\pi r/a} \\ &= f + 2(u_0 - f)\sum_{n=1}^{\infty}(-1)^{n+1}e^{-kn^2\pi^2 t/a^2}, \end{aligned}$$

at least for $t > 0$. Notice that the series in (12.25) does not converge for $t = 0$. The result is plotted in Figure 2. Clearly we need to cook the turkey for a little more than 5 hours.

12.5. Interchanging Infinite Sums and Limits

Let's return to the computation in (12.25). We skipped a step. The computation should read

(12.26)
$$\begin{aligned} u(0,t) &= \lim_{r\to 0} u(r,t) \\ &= f + 2(u_0 - f)\lim_{r\to 0}\sum_{n=1}^{\infty}(-1)^{n+1}e^{-kn^2\pi^2 t/a^2}\frac{\sin(n\pi r/a)}{n\pi r/a} \\ &= f + 2(u_0 - f)\sum_{n=1}^{\infty}(-1)^{n+1}e^{-kn^2\pi^2 t/a^2}\lim_{r\to 0}\frac{\sin(n\pi r/a)}{n\pi r/a} \\ &= f + 2(u_0 - f)\sum_{n=1}^{\infty}(-1)^{n+1}e^{-kn^2\pi^2 t/a^2}, \end{aligned}$$

Between the second and third lines in (12.26) we have interchanged a limit and an infinite sum. This should never be done without checking that it is legitimate. For $t > 0$ it is legitimate because the exponential terms decrease so rapidly, but for $t = 0$ it isn't.

Let's concentrate on the sum and limit terms and do the computation in both orders for $t = 0$. First, using (12.22) we have

$$\lim_{r \to 0} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n\pi r/a)}{n\pi r/a} = \lim_{r \to 0} \frac{1}{2} = \frac{1}{2}. \tag{12.27}$$

If we interchange the sum and the limit, we get

(12.28)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \lim_{r \to 0} \frac{\sin(n\pi r/a)}{n\pi r/a} = \sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - \cdots ,$$

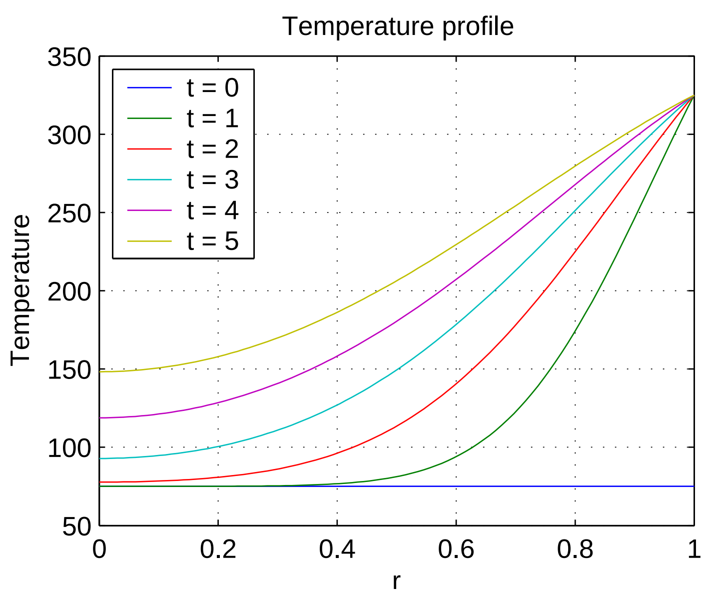
which is a divergent series. Thus computing the limit of the sum we get the number $1/2$, while the sum of the limits leads to a divergent series. This is just one of the many strange things that can happen when we interchange a sum and a limit.

Although the series in (12.28) does not converge, the mathematician Leonhard Euler insisted that the sum is $1/2$. In his defense we should add that no rigorous definition of convergence existed at the time. However, setting the sum equal to $1/2$ makes the results in (12.27) and (12.28) equal. It also conveniently gives the correct answer $u(0, 0) = u_0$ in the last line of (12.25)! It is amazing how often using $1/2$ as the sum of this strange, divergent series leads to a correct result.

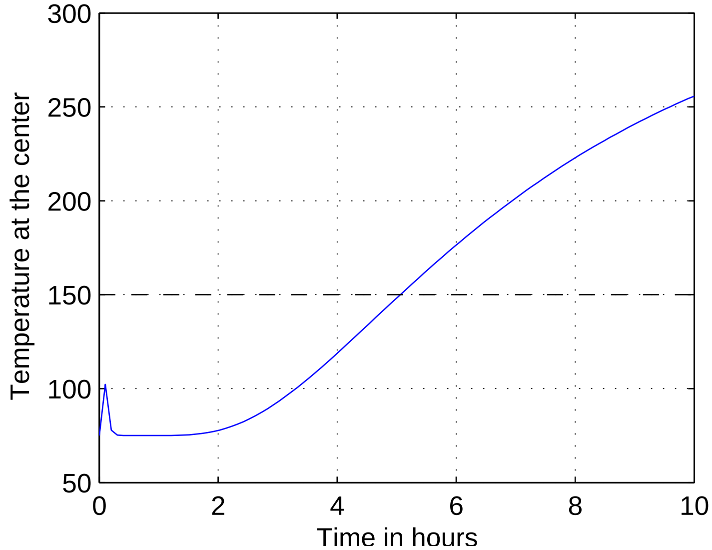
Here is a theorem giving conditions under which a limit and an infinite sum can be interchanged.

Theorem. Suppose that $f_n(x)$ is a sequence of functions defined on an interval $I = [a, b]$, and that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I. Suppose in addition that $x_0 \in I$, and that $\lim_{x \to x_0} f_n(x)$ exists for each n. Then

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} f_n(x).$$



Temperature profile
t = 0
t = 1
t = 2
t = 3
t = 4
t = 5
Temperature
350
300
250
200
150
100
50
0
0.2
0.4
0.6
0.8
1
r



300
250
200
150
100
50
Temperature at the center
0
2
4
6
8
10
Time in hours