An Introduction to Partial Differential Equations in the Undergraduate Curriculum

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LECTURE 14
Playing the Timpani: Vibrations of a circular membrane

Outline of Lecture

14.1. Observations about timpani playing

A timpani (or kettledrum) is a percussive instrument consisting of a circular drumhead (usually plastic, but in older times, an animal skin) that is tautly stretched over a metal bowl. As a first approximation, the vibrations of the timpani's drumhead can be modelled by the wave equation,

$$u_{tt} = c^2 \nabla^2 u,$$

where c is the speed of waves travelling on the drumhead. The constant c is directly related to the tension of the drumhead and the corresponding pitch that is generated by hitting the drumhead with a mallet, and can be adjusted using a foot pedal. The characteristic sound of the timpani is determined by its *vibrational modes* and their corresponding frequencies.

Any timpani player will tell you that the proper place to strike the drumhead is not the center of the drumhead, but rather a spot somewhere about one-sixth of the diameter away from the edge of the drumhead. The most common timpanis have a diameter between 23 to 29 inches, so that means striking the timpani about 4 to 5 inches in from the edge of the drumhead. Striking the drumhead in the center produces a sound that is somewhat hollow. In this lecture, we will give some mathematical explanations for why this occurs.

14.2. Solution to the wave equation via separation of variables

As we are describing the vibrations of a circular membrane, it is convenient to use polar coordinates. Let the displacement of the membrane be $u = u(r, \theta, t)$, in which case the wave equation can be more explicitly written as

$$u_{tt} = c^2 \left[u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right]. \tag{14.1}$$

(Refer to Prof. Tolosa's Lecture 11.) Since the drumhead is tautly held down, we impose Dirichlet conditions at the boundary of the drumhead:

$$u(a, \theta, t) = 0,$$

where a is the radius of the drumhead. Later on, we will also need two less-obvious boundary conditions: that the displacement is finite at the origin ($r = 0$), and that the displacement is a 2π-periodic function in θ (refer again to Lecture 11). To obtain u explicitly, we would also need to know the initial displacement, $u(r, \theta, 0)$, and initial velocity, $u_t(r, \theta, 0)$. However, our goal is simply to understand more about the vibrational modes (eigenfunctions) of the circular membrane, so these aren't necessary; just obtaining the general solution using separation of variables will suffice.

Let's make the usual separation of variables ansatz that $u(r, \theta, t) = R(r)H(\theta)T(t)$, and substitute this into (14.1). The result is a set of three ordinary differential equations,

$$H'' + n^2 H = 0, \tag{14.2a}$$

$$T'' + c^2 \lambda^2 T = 0, \tag{14.2b}$$

$$r^2 R'' + rR' + (\lambda^2 r^2 - n^2)R = 0, \tag{14.2c}$$

where the separation constants are λ^2 and n^2.

As we saw in Lecture 11, the first equation (14.2a) has 2π-periodic solutions only if n is an integer, so we restrict $n = 0, 1, 2, \ldots$, and define

$$H_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta).$$

The second equation (14.2b) has the general solution

$$T(t) = \alpha \cos(c\lambda t) + \beta \sin(c\lambda t),$$

assuming $\lambda \geq 0$, which means that the frequency of any particular vibrational mode is $2\pi/(c\lambda)$.

The third ODE (14.2c) is related to Bessel's equation of order n (not to be confused with the order of the differential equation itself) and can be written in Sturm-Liouville form:

$$\frac{d}{dr} \left[rR'(r) \right] - \frac{n^2}{r} R(r) + \lambda^2 r R(r) = 0. \tag{14.3}$$

Please refer to equation (1.3) from Lecture 9. This equation should be solved over the interval $0 \leq r \leq a$ subject to the conditions that $R(0)$ be finite and $R(a) = 0$. Because we are solving this equation on the interval $0 \leq r \leq a$ and the functions r and r^{-1} are not positive at the endpoints of this interval, this boundary-value problem is not a regular Sturm-Liouville eigenvalue problem but rather a singular Sturm-Liouville eigenvalue problem. Furthermore, Bessel's equation is linear and has a regular singular point at $r = 0$, so we expect that at least one of the linearly independent solutions to this equation has a singularity at $r = 0$.

Nevertheless, this eigenvalue problem still has a complete set of orthogonal eigenfunctions. From (14.3), we see that the weight function is r, so inner products should be computed as integrals over the interval $0 \le r \le a$ with an extra factor of r. (This extra factor accounts for the fact that the area of a circle is proportional to the square of its radius.)

The general solution to (14.2c) is

$$R(r) = CJ_n(\lambda r) + DY_n(\lambda r),$$

where $J_n(z)$ and $Y_n(z)$ are Bessel's functions of the first and second kind, respectively. For more information, refer to the handout on Bessel functions, and Section 14.5. Bessel's functions of the second kind have singularities at $r = 0$, so for $R(0)$ to remain finite we must choose $D = 0$. The other boundary condition $R(a) = 0$ requires that

$$CJ_n(\lambda a) = 0.$$

We do not want $C = 0$ or we will have the trivial solution, so we require $J_n(\lambda a) = 0$. Even though the zeros of $J_n(z)$ are not evenly spaced like the sine and cosine functions, they can be calculated. Let us denote z_{mn} to be the mth positive root of $J_n(z)$, as shown in the figure below. Table 14.2 lists some values of z_{mn}.

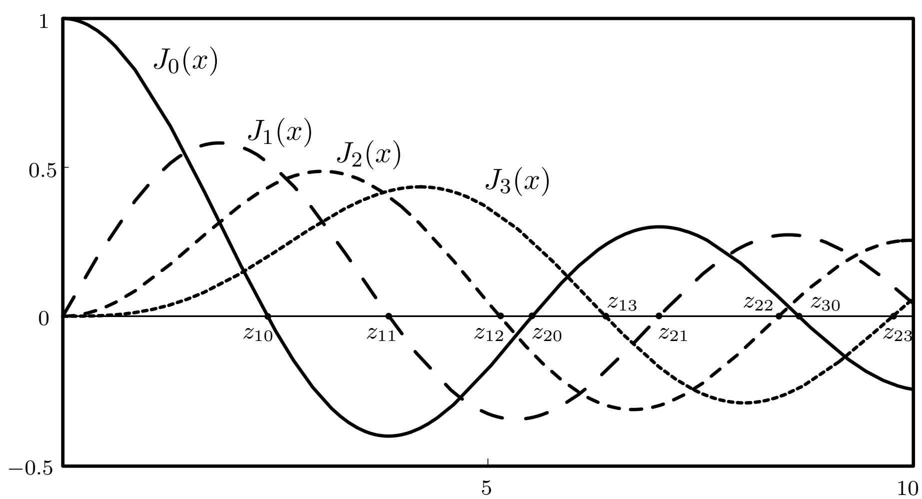

Figure 14.1. Graphs of a few Bessel functions of the first kind with their zeros marked.

z_{mn}	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$m = 1$	2.4048	3.8317	5.1356	6.3802
$m = 2$	5.5201	7.0156	8.4172	9.7610
$m = 3$	8.6537	10.173	11.620	13.015
$m = 4$	11.792	13.324	14.796	16.223

Table 14.2. Approximate locations of the zeros of Bessel functions of the first kind: z_{mn} denotes the mth positive root of $J_n(z)$.

Armed with the zeros of $J_n(z)$, we define

$$\lambda_{mn} = \frac{z_{mn}}{a},$$

for $m = 1, 2, 3\ldots$, and $n = 0, 1, 2, \ldots$. The product solutions that describe the vibrational modes of the circular drumhead are therefore

$$u_{mn}(r,\theta,t) = J_n\left(\frac{z_{mn}r}{a}\right)\left[A_{mn}\cos(n\theta) + B_{mn}\sin(n\theta)\right]\left[\alpha_{mn}\cos(c\lambda_{mn}t) + \beta_{mn}\sin(c\lambda_{mn}t)\right].$$

To solve explicitly for u, one needs to create a proper superposition (some of the constants above may be superfluous) of product solutions and determine constants from initial conditions using orthogonality conditions. This is left as an exercise to the reader in Problems 2 and 3.

Because of the trigonometric identity

$$\alpha\cos\phi + \beta\sin\phi = \sqrt{\alpha^2+\beta^2}\cos(\phi+\phi_0),$$

where ϕ_0 is a phase shift, we recognize that the product solution u_{mn} above can be written in a more compact form. If we are only after the qualitative behavior of each vibrational mode, we can ignore the angular and temporal phase shift by defining

$$v_{mn}(r,\theta,t) = J_n\left(\frac{z_{mn}r}{a}\right)\cos(n\theta)\cos\left(\frac{cz_{mn}t}{a}\right),$$

for $m = 1, 2, 3\ldots$, and $n = 0, 1, 2, \ldots$. Plots of these vibrational modes appear at the end of this handout. Each of these vibrational modes has a frequency of $2\pi a/(cz_{mn})$. When the timpani is struck, it is the combination of all of the vibrational modes and their corresponding frequencies that contributes to its characteristic sound (timbre).

14.3. Axisymmetric versus nonaxisymmetric cases

Now, we are in a position to discuss the differences between striking the timpani in the center and off center. If the timpani is struck exactly in the center, and the impact of the mallet creates an initial displacement and velocity that is axisymmetric (depending only on r), then one can assume that the solution remains axisymmetric for all time. In other words, if $u(r,\theta,0)$ and $u_t(r,\theta,0)$ are independent of θ, then $u(r,\theta,t)$ is also independent of θ.

Separation of variables in the axisymmetric case is a bit easier, with the general solution taking the form

$$u(r,t) = \sum_{m=1}^{\infty} J_0\left(\frac{z_{m0}r}{a}\right)\left[\alpha_m\cos(c\lambda_{m0}t) + \beta_m\sin(c\lambda_{m0}t)\right].$$

Therefore, we see that only a subset of the vibrational modes from the nonaxisymmetric case are excited (those corresponding to $n = 0$). The allowable vibrational frequencies are similarly limited, which is why the sound of the timpani is hollow when the timpani is struck in the center.

The same conclusion can be obtained by looking at the vibrational modes of the nonaxisymmetric case (at the end of this handout). Notice that for $n \geq 1$, each vibrational mode has node lines (i.e. curves on the drumhead for each vibrational mode that experience no displacement) that pass through the center. So, if the timpani is struck at the center, even if the initial velocity and displacement are nonaxisymmetric, none of the vibrational modes for $n \geq 1$ can be excited.

The purpose of striking the timpani about one-sixth of the diameter away from the edge of the drumhead is to excite the $m = 1$, $n = 1$ vibrational mode. It turns out that this vibrational mode, along with the other *preferentially excited modes*, produce the most pleasing sound. These observations can be found in Lord Rayleigh's classic book "The Theory of Sound" (1877).

14.4. References and extensions

- In pratice, the actual sound that we hear from a timpani is different from that predicted by the analysis above. There are several important factors that have been neglected. First, and most importantly, the motion of the timpani is damped, which changes the vibrational modes and their frequencies. (Refer to Lecture 7 by Profs. Vajiac and Tolosa.) Sound waves travelling from the timpani to your ear also experience damping. Second, there are small nonlinear effects (such as surface tension), which exert their own preference for certain modes by transferring energy from one vibration mode to another. Third, the tension of the drumhead is not uniform across the entire drumhead (c is not really constant). Furthermore, there is a coupling between the vibrations of the membrane, the vibrations of the membrane bowl, and the vibrations of the air particles that eventually reach your ear. To learn more about musical acoustics, try "The Acoustical Foundations of Music" by John Backus and "Fundamentals of Musical Acoustics" by Arthur Benade.
- Consider the wave equation on an arbitrary planar domain Ω:

  $$u_{tt} = c^2\nabla^2 u.$$

  Regardless of what Ω looks like, one can still separate the temporal part of the solution from the spatial part by assuming $u = T(t)\Phi$, where Φ is a function of all the spatial variables (for example, x, y, z, or r, θ). Substituting this ansatz into the wave equation, we obtain

  $$T''(t) + c^2\lambda^2 T(t) = 0$$

  $$\nabla^2\Phi + \lambda^2\Phi = 0. \tag{14.4a}$$

  Equation (14.4) is known as Helmholtz's equation. The domain Ω and the boundary conditions specified on $\partial\Omega$ determine the eigenfunctions Φ and the eigenvalues λ^2. (See Problems 1 and 5.) If we are thinking about vibrating membranes like drumheads, then the eigenfunctions are the vibrational modes and the eigenvalues determine their associated frequencies.

  One very natural question to ask is, whether the *inverse problem* is solvable. In other words, if we are given the vibrational frequencies of a drumhead, can we determine the shape of the membrane? Marc Kac posed and answered this question in a brilliantly written paper "Can one hear the shape of a drum?" (Amer. Math. Monthly 73, 1966, no. 4, pages 1–23).
- Separation of variables can be used to determine these vibrational modes in situations where the geometry of the membrane is simple enough, but most real-world problems do not involve simple geometries. What can be done in these situations? For complicated domains composed of simpler pieces (like a square joined with a semicircle), one can analytically obtain bounds on the eigenvalues. (See "Partial

Differential Equations: An Introduction" by Walter Strauss.) If the shape of the membrane is very close to one for which separation of variables works, *regular perturbation techniques* can give analytic approximations to the eigenfunctions and eigenvalues. (See "Perturbation Methods" by E. J. Hinch.) And of course, numerical methods can be used, as we learned from Prof. Arnold in yesterday's lecture. Complicated geometries in two- or three-dimensions are usually best handled using *finite element numerical methods*.

14.5. Properties of Bessel functions

First, we show how equation (14.2c) can be transformed to Bessel's equation. Let $y(x) = R(r)$ where $x = \lambda r$. Using the chain rule,

$$\frac{dR}{dr} = \frac{dy}{dx}\frac{dx}{dr} = \frac{dy}{dx}\lambda.$$

This change of variables changes (14.2c) to Bessel's equation of order n (which is not necessarily integral),

$$x^2 y'' + xy' + (x^2 - n^2)y(x) = 0. \tag{14.5}$$

This linear ODE has a regular singular point at $x = 0$. Therefore, the two linearly independent solutions to this equation can be obtained using a Frobenius series expansion,

$$y(x) = x^\alpha \sum_{k=0}^{\infty} a_k x^k, \tag{14.6}$$

where α is some real number to be determined. Substituting (14.6) and its derivatives into (14.5), we obtain

$$\sum_{k=0}^{\infty} a_k(k+\alpha)(k+\alpha-1)x^{k+\alpha} + \sum_{k=0}^{\infty} a_k(k+\alpha)x^{k+\alpha} + \sum_{k=0}^{\infty} a_k x^{k+\alpha+2} - n^2 \sum_{k=0}^{\infty} a_k x^{k+\alpha} = 0.$$

We wish to identify the coefficient of each power of x, so we renumber the third term above so that all the powers of x match, then we take out the $k = 0$ and $k = 1$ terms out of all remaining terms, and regroup.

$$a_0(\alpha^2 - n^2)x^\alpha + a_1((\alpha+1)^2 - n^2)x^{\alpha+1} + \sum_{k=2}^{\infty} \left[a_k((k+\alpha)^2 - n^2) + a_{k-2}\right] x^{k+\alpha} = 0$$

Since the right-hand side of the equation above is zero, the only way for the equation to be true is for the coefficient of every power of x to also be zero. Therefore,

$$a_0(\alpha^2 - n^2) = 0 \tag{14.7a}$$

$$a_1((\alpha+1)^2 - n^2) = 0 \tag{14.7b}$$

$$a_k((\alpha+k)^2 - n^2) = -a_{k-2} \qquad \text{for } k = 2, 3, \dots. \tag{14.7c}$$

Since α is chosen so that x^α is the first term of the series, we should not have a_0 be zero, so the only way for (14.7a) to be satisfied is if $\alpha^2 - n^2 = 0$. This equation is known as the *indicial equation*, and it implies that $\alpha = \pm n$. Notice that this choice of α necessarily implies that $a_1 = 0$, and since the recurrence relation (14.7c) links every other coefficient

a_k together, $a_1 = 0$ implies that $a_k = 0$ for all odd k. Therefore, each root of the indicial equation leads to a linearly independent solution of Bessel's equation except in the case when n is an integer.

When n is an integer, choosing $\alpha = |n|$ gives one of the linearly independent solutions. However, choosing $\alpha = -|n|$ does not give a second linearly independent solution because the recurrence relation (14.7c) leads to a division by zero when calculating $a_{2|n|}$. In this situation, the second linearly independent solution can be determined by reduction of order. (See "Advanced Mathematical Methods for Scientists and Engineers" by Bender and Orszag for a complete treatment of Frobenius series.)

For the purposes of the current discussion, just keep in mind these two facts. First, Bessel's functions of the first and second kind are defined as the solutions to a linear second-order differential equation in the same way that the sine and cosine functions are the two linearly independent solutions of the linear oscillator equation. They may not be as commonly used as the sine and cosine functions, but their values and zeros are just as calculated or tabulated, especially with a computer. Furthermore, many of the trigonometric identities have analogues with their Bessel functions cousins.

Second, the indicial equation tells us the leading behavior of the Bessel functions near $x = 0$. The Bessel function of the first kind is defined as the linearly independent solution corresponding to $\alpha = |n|$, so $J_n(x)$ behaves like $x^{|n|}$ near $x = 0$ (since that is the first term of the power series). For non-integral n, the Bessel function of the second kind of order n, $Y_n(x)$, behaves like $x^{-|n|}$ near $x = 0$. And for integral n, the $Y_n(x)$ also has $x^{-|n|}$ as its first power series term, but has a logarithmic term ($\ln x$) as well. This means that for any n, the Bessel function of the second kind always diverges as $x \to 0$. As a result, most problems in polar or spherical coordinates whose domains include the origin usually involve discarding Bessel functions of the second kind.

14.6. Challenge Problems for Lecture 14

Problem 1. Determine the vibrational modes of a timpani with a rectangular drumhead. In other words, find the product solutions for the wave equation

$$u_{tt} = c^2 \nabla^2 u$$

in the rectangle $0 \leq x \leq L$ and $0 \leq y \leq W$, subject to homogeneous Dirichlet boundary conditions along all four sides of the rectangle.

Problem 2 (Axisymmetric case)**.** Suppose $u(r, \theta, 0) = f(r)$ is the initial displacement of a timpani's drumhead (with radius a), and that the drumhead has zero initial velocity. Solve for the displacement, $u(r, \theta, t)$, showing how all constants are calculated from f.

Problem 3 (Nonaxisymmetric case)**.** Suppose $u(r, \theta, 0) = f(r, \theta)$ is the initial displacement of a timpani's drumhead (with radius a), and that the drumhead has zero initial velocity. Solve for the displacement, $u(r, \theta, t)$, showing how all constants are calculated from f.

Problem 4. Once you've done either of the previous problems, create an animation of your solution to the wave equation for an initial condition of your choice.

Problem 5. Determine and discuss the properties of the vibrational modes of an annular membrane. (**Hint:** You'll need to use both $J_n(z)$ and $Y_n(z)$.)