An Introduction to Partial Differential Equations in the Undergraduate Curriculum

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LECTURE 3
Laplace's Equation & Harmonic Functions

1.1. Outline of Lecture

- Laplace's Equation and Harmonic Functions
- The Mean Value Property
- Dirichlet's Principle
- Minimal Surfaces

1.2. Laplace's Equation and Harmonic Functions

Let Ω be an open subset of $\mathbb{R}^n = \{(x_1, \ldots, x_n) \,|\, x_i \in \mathbb{R}\}$ and suppose $u : \Omega \to \mathbb{R}$ is given. Recall that the gradient of u is defined as

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right).$$

The **Laplacian of** u, denoted Δu or $\nabla^2 u$ is defined as

$$\Delta u = \mathrm{div}\,(\nabla u) = u_{x_1 x_1} + \cdots + u_{x_n x_n}.$$

Laplace's equation on the domain Ω is

$$\Delta u = 0, \qquad (x_1, \ldots, x_n) \in \Omega.$$

Solutions to Laplace's equation on Ω are called **harmonic functions** on Ω.

In one dimension harmonic functions are linear. Things are not so simple in higher dimensions. Let's consider some examples.

(a) **Steady-state Fluid Flow**. In fluid mechanics one is interested in the velocity field $\vec{v} = \vec{v}(x, y, z, t)$ of a given fluid. If the flow is steady, then the velocity field is independent of time t. If the flow is irrotational (i.e., curl $\vec{v} = 0$), then $\vec{v} = -\nabla u$ for some scalar function u (called the **velocity potential**). If the flow is incompressible (e.g., constant density), then div $\vec{v} = 0$. But now div $\vec{v} = \text{div}(-\nabla u)$. Consequently $\Delta u = \text{div}\nabla u = 0$.

Thus the velocity potential for an incompressible irrotational fluid is harmonic. This is a very important result in the theory of fluid dynamics.

(b) **Electrostatics.** Maxwell's equations govern the interaction between electric and magnetic fields. In the static case the equations for the electric field and magnetic field decouple and the electric field $\vec{E}$ is governed by the two equations

$$\text{curl } \vec{E} = \vec{0} \qquad \text{div}\vec{E} = 4\pi\rho,$$

where ρ is the charge density. Since $\vec{E}$ is curl free it follows that $\vec{E} = -\nabla\Phi$ for some scalar function Φ (called the **electric potential**). Substituting this into the second equation yields $\Delta\Phi = -4\pi\rho$. Thus, in any charge free region $\rho = 0$ and $\Delta\Phi = 0$.

In words, the electric potential in a charge free region is harmonic. Thus one can find the electric field by solving the PDE and taking the gradient of the solution.

(c) **Analytic Functions.** Let $z = x + iy$. An analytic function

$$f(z) = u(x, y) + i\,v(x, y)$$

satisifes the Cauchy Riemann equations:

$$u_x = v_y \qquad u_y = -v_x.$$

It follows that $u_{xx} = (v_y)_x = (v_x)_y = (-u_y)_y = -u_{yy}$ or

$$u_{xx} + u_{yy} = 0.$$

Similarly, $\Delta v = 0$. Thus the real and imaginary parts of an analytic function are harmonic! This gives us an endless source of interesting harmonic functions. For example,

- $f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy)$. Thus $x^2 - y^2$ and $2xy$ are harmonic on $\mathbb{R}^2$.
- $f(z) = z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta) = r^n \cos n\theta + i\, r^n \sin n\theta$. Thus $r^n \cos n\theta$ and $r^n \sin n\theta$ are harmonic. What are these functions in terms of x and y?

- $f(z) = \ln z = \ln|z| + i\,\mathrm{Arg}(z) = \ln\sqrt{x^2+y^2} + i\arctan(y/x)$. Thus $\ln\sqrt{x^2+y^2}$ and $\tan^{-1}(y/x)$ are harmonic. For what regions are they harmonic?

- (more exotic) $f(z) = \frac{e^{-z^2}}{z^2} + z^4$. It's not obvious what the real and imaginary parts are but we can readily visualize them using Maple.

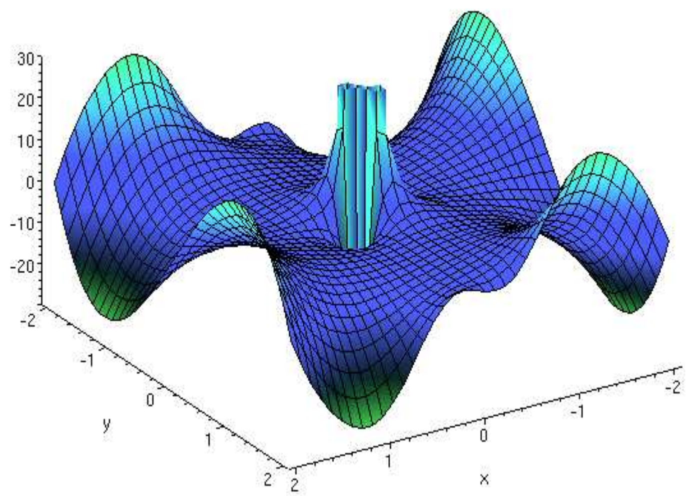

Figure 1. The real part of $f(z) = \frac{e^{-z^2}}{z^2} + z^4$.

See the Maple worksheet for more plots of these surfaces. Notice that they all have a sense of flatness to them. They bend and curve, but in a curious way. Notice all the local maxima or minima occur on the boundary of the surface. There's another situation where you may have observed similar "flat" surfaces: soap films.

1.3. As Flat as Possible: Take 1 The Mean Value Property

Question Suppose $g : \partial B(0,1) \to \mathbb{R}$ is given and $u : \overline{B(0,1)} \to \mathbb{R}$ is a surface that agrees with g on $\partial\Omega$ and is **as flat as possible**. What should the value $u(0,0)$ be?

One Answer: $u(0,0) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} g(\theta)\,d\theta.$
That is u should be the average of all the boundary data.

More generally, suppose Ω is an open subset of $\mathbb{R}^2$, $g : \partial\Omega \to \mathbb{R}$ is given, and $u : \overline{\Omega} \to \mathbb{R}$ satisfies $u = g$ on $\partial\Omega$ and u is *as flat as possible.* Can we determine u?

Motivated by the first question, let's suppose

$$u(P) = \frac{1}{2\pi r}\int_{\partial B(P,r)} u\,ds, \tag{1.1}$$

for each $P \in \Omega$ and $B(P,r) \subset \Omega$. In words, suppose u at P is the average of u over any circle centered at P. Could such a function exist?

Let $P = (x_0, y_0)$ and consider the following figure:

Then

$$\begin{aligned}
u(x_0,y_0) &= \frac{1}{2\pi r}\int_{\partial B(P,r)} u(x,y)\,ds \\
&= \frac{1}{2\pi r}\int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\,r d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\,d\theta.
\end{aligned}$$

Notice that the left-hand side is constant while the right-hand side is a function of r. We now exploit this by taking the derivative with respect to r (remembering the chain rule):

$$\begin{aligned}
0 &= \frac{1}{2\pi}\int_0^{2\pi} u_x \cos\theta + u_y \sin\theta \, d\theta \\
&= \frac{1}{2\pi r}\int_0^{2\pi} \left(u_x \cos\theta + u_y \sin\theta\right) \, r d\theta \\
&= \frac{1}{2\pi r}\int_{\partial B(P,r)} \nabla u \cdot \nu \, ds \\
&= \frac{1}{2\pi r}\int_{B(P,r)} \operatorname{div}\left(\nabla u\right) \, dy\, dx \\
&= \frac{1}{2\pi r}\int_{B(P,r)} \Delta u \, dy\, dx.
\end{aligned}$$

The second to last equality follows from the divergence theorem.

Aha! If (1.1) holds then

$$0 = \int_{B(P,r)} \Delta u \, dy\, dx$$

for all $r > 0$ with $B(P, r) \subset \Omega$. If this holds for all $P \in \Omega$, then

$$\boxed{\Delta u = 0, \quad \text{for each } P \in \Omega.}$$

We have established the following theorem:

Theorem 1 (Mean-Value Property)
The function u is harmonic if and only if

$$u(P) = \frac{1}{2\pi r}\int_{\partial B(P,r)} u \, ds = \frac{1}{\pi r^2}\int_{B(P,r)} u \, dxdy,$$

for each $P \in \Omega$ and $B(P, r) \subset \Omega$.

This theorem answers our question about u. To find the function u that equals g on $\partial\Omega$ and is "as flat as possible"(in the sense we chose) we can solve the boundary value problem:

$$\begin{cases} \Delta u = 0, & (x, y) \in \Omega, \\ u = g, & (x, y) \in \partial\Omega. \end{cases} \tag{1.2}$$

This explains the apparent "flatness"of the harmonic surfaces we've looked at. They can bend but only in such a way as to always preserve the mean value property.

This is a rather curious way to derive a PDE. Starting from the pointwise assumption that u is always the average of its neighboring values we arrived at the conclusion that u must be harmonic! There is nothing special about 2D here, and in fact, the same result holds in $\mathbb{R}^n$, now taking the average over a sphere or hypersphere ($n \geq 4$).

1.4. As Flat As Possible: Take 2 Dirichlet's Principle

Let's consider another approach to the question of finding a surface u that agrees with g on $\partial\Omega$ and is as flat as possible. Rather than define a pointwise constraint, let's assign a numerical measure of flatness to each surface u that agrees with g on $\partial\Omega$:

$$E[u] = \frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dy \, dx.$$

This integral is often referred to as the **Dirichlet energy** integral, in analogy with the kinetic energy $\frac{1}{2}mv^2$. At this point it is a somewhat arbitrary choice, you could choose your own numerical measure and perform similar computations.

Notice that if $E[u] = 0$, then $|\nabla u| = 0$ in Ω, in which case u is constant, quite flat. On the other hand, if g is nonconstant and $u = g$ on $\partial\Omega$ then $E[u] > 0$ and we can ask what is true for functions u that minimizes this quantity?

Let u minimize E over

$$\mathcal{A} = \{w : \overline{\Omega} \to \mathbb{R} : w = g \text{ on } \partial\Omega\}.$$

For ϕ smooth, with $\phi = 0$ on $\partial\Omega$ notice that $u + \phi \in \mathcal{A}$ and

$$E[u] \leq E[u + \phi].$$

In fact, for each such ϕ we can define a map $i_\phi : \mathbb{R} \to \mathbb{R}$

$$i_\phi(\epsilon) = E[u + \epsilon\phi].$$

Then u minimizes E implies that the function i_ϕ has a minimum at $\epsilon = 0$, or $i'_\phi(0) = 0$. But $i_\phi : \mathbb{R} \to \mathbb{R}$ is an honest single variable function whose derivative we can compute:

$$\begin{aligned}
i(\epsilon) &= E[u+\epsilon\phi] \\
&= \frac{1}{2}\int_\Omega |\nabla u + \epsilon\nabla\phi|^2\,dV \\
&= \frac{1}{2}\int_\Omega (\nabla u + \epsilon\nabla\phi)\cdot(\nabla u + \epsilon\nabla\phi)\;dV \\
&= \frac{1}{2}\int_\Omega |\nabla u|^2 + 2\epsilon\,\nabla u\cdot\nabla\phi + \epsilon^2|\nabla\phi|^2\,dV.
\end{aligned}$$

Thus

$$i'(\epsilon) = \int_\Omega \nabla u\cdot\nabla\phi\,dV + \epsilon\int_\Omega |\nabla\phi|^2\,dV,$$

or, letting $\epsilon = 0$ yields,

$$0 = \int_\Omega \nabla u\cdot\nabla\phi\,dV = \int_\Omega (\Delta u)\,\phi\,dV.$$

Thus, u minimizes E over $\mathcal{A}$ only if

$$\int_\Omega (\Delta u)\,\phi\,dV = 0 \qquad \text{for all } \phi.$$

Since ϕ is arbitrary, it follows that

$$\Delta u = 0 \qquad (x,y)\in\Omega.$$

We have shown that harmonic functions minimize the Dirichlet energy. This is known as **Dirichlet's principle**. Notice that two completely different approaches have led to the same PDE. This second method is known as the **Calculus of Variations** and is a very active area of current research. Choosing a different numerical measure E will yield a different PDE.

1.5. Minimal Surfaces & Harmonic Functions

Are the soap film surfaces harmonic functions? A soap surface is flat since surface tension acts to minimize the surface area. Thus they minimize

$$SA[u] = \int_\Omega \sqrt{1+|\nabla u|^2}\,dxdy, \tag{1.3}$$

over all functions that agree with g on $\partial\Omega$.

Using the Taylor approximation $\sqrt{1+x}\approx 1+\frac{x}{2}$ we can rewrite this as

$$SA[u] = \int_\Omega \sqrt{1+|\nabla u|^2}\,dxdy \approx \int_\Omega 1 + \frac{1}{2}|\nabla u|^2\,dxdy.$$

For purposes of minimizing the latter integral we can ignore the constant term. Thus we see that for $|\nabla u|^2$ small, minimizing the surface area is equivalent to minimizing the Dirichlet energy integral! Thus soap surfaces are not harmonic, but close.

Performing the computations of the last section with the surface area integral (1.3) yields the **minimal surface equation**:

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

As an exercise, verify that in two two-dimensional case this can be expressed as

$$(1 + u_y^2)u_{xx} + 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0. \tag{1.5}$$

1.6. Challenge Problems for Lecture 3

Problem 1. Find the harmonic functions defined by the real and imaginary part of

(a) $f(z) = z^3$
(b) $f(z) = e^z$

Problem 2. Show that in the two dimensional case the minimal surface equation (1.4) is the same as (1.5).

Problem 3. Recall the mean value property states that u is harmonic if and only if

$$u(P) = \frac{1}{2\pi r}\int_{\partial B(P,r)} u\, ds = \frac{1}{\pi r^2}\int_{B(P,r)} u\, dxdy$$

for each $P \in \Omega$ and $B(P,r) \subset \Omega$. In class we proved the first equality. Prove the second equality holds by expressing the integral over $B(P,r)$ in terms of $\partial B(P,s)$ where $0 \le s \le r$.

Problem 4. Use the Mean Value Property to prove the Maximum Principle:

Theorem Let Ω be a bounded domain. If u is harmonic on Ω then

(a) $\max\limits_{\overline{\Omega}} u = \max\limits_{\partial\Omega} u$.
(b) If Ω is connected and there exists $\vec{x_0} \in \Omega$ such that

$$u(\vec{x_0}) = \max_{\partial\Omega} u,$$

then u is constant.

This second statement is known as the **Strong Maximum Principle** and is a very important tool in the theory of PDE. First prove part (b), then note that (a) follows from (b).

Problem 5. Use the Maximum Principle to prove that the Dirichlet problem

$$\begin{cases} \Delta u = f, & (x,y) \in \Omega, \\ u = g, & (x,y) \in \partial\Omega, \end{cases} \tag{1.6}$$

has at most one solution. Hint: Consider the difference of two solutions.