An Introduction to Partial Differential Equations in the Undergraduate Curriculum

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LECTURE 6
The Convergence of Fourier Series

6.1. Outline of Lecture

- Back in the 19th Century...
- Fourier Series: The Basics
- Examples
- Other Intervals
- Coefficients and their Decay
- Linear Algebra to the Rescue: L^2
- The Many Faces of Convergence
- The Way Things Are

6.2. Back in the 19th Century...

'All is number', said Pythagoras, by which he meant that the sounds we hear are built from simpler, more basic frequencies. At the start of the 19th century Fourier explained how actually to decompose a function into simple sine waves, a process that relates neatly to the old Greek assertion. Fourier's method was very simple to apply, and very difficult to justify rigorously, so much so that many of his contemporaries did not believe his results. As you have seen in the previous lecture, he used his method to offer a solution to the problem of heat propagation on a metal bar. Here we will be concerned with Fourier series themselves, their behavior, aspects of their convergence, etc.

6.3. Fourier Series: The Basics

To fix ideas we will work over the interval $[-\pi, \pi]$, and deal with variations later. Fourier's setup is that given a function f defined over that interval, it is possible to write it as a (possibly infinite) linear combination of the 'pure' waves 1, $\cos(x)$, $\cos(2x)$, $\dots$, $\sin(x)$, $\sin(2x)$, etc, so that

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx). \tag{6.1}$$

Soon it will become clear why we have divided the zeroth coefficient by 2.

For the moment let's believe (6.1), and see how to compute the coefficients A_n, B_n. The key here are the following formulas, called *orthogonality relations.*

Exercise 1. Show that if m, n are non-negative integers, and $m \neq n$, then

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = 0,$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0.$$

Also, even if we drop the restriction $m \neq n$,

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0.$$

Exercise 2. Show that if n is a positive integer, then

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi.$$

Without worrying about rigor, let's follow Fourier and obtain formulas for the coefficients. The coefficient A_0 is the simplest to find: integrate (6.1) from $-\pi$ to π to get

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{A_0}{2}\,dx + \sum_{n=1}^{\infty} \left\{ A_n \int_{-\pi}^{\pi} \cos(nx)\,dx + B_n \int_{-\pi}^{\pi} \sin(nx)\,dx \right\}.$$

The series on the right vanishes, and we find that

$$A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx.$$

We do the same thing to compute, say, B_m, except that first we multiply (6.1) through by $\sin(mx)$. We get

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx =$$
$$\int_{-\pi}^{\pi} \frac{A_0}{2}\sin(mx)\,dx +$$
$$\sum_{n=1}^{\infty} A_n \int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx + B_n \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx.$$

What is important to notice is that *all* of the integrals on the right side vanish, except for the one multiplying B_m. The equation for B_m becomes

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx \qquad m = 1, 2, 3, \ldots. \tag{6.2}$$

Likewise the formula for A_m is

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \qquad m = 0, 1, 2, \ldots. \tag{6.3}$$

(Because A_0 appears in (6.1) divided by 2, the above formula for A_m also works for A_0.)

Formulas (6.2) and (6.3) allow us to compute the Fourier coefficients of f. We postpone any discussion about convergence until Section 6.6.

One last important point: Even though f is defined only on $[-\pi, \pi]$, the right-hand side of (6.1) is 2π-periodic, so we could view f as being defined over the whole line, but 2π-periodic as well.

6.4. Examples

For our first example we consider $f(x) = x$ over the interval $[-\pi, \pi]$. Clearly $A_0 = 0$, and for $n \geq 1$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = \frac{1}{\pi}\left\{ \frac{x\sin(nx)}{n}\bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin(nx)}{n}\,dx \right\} = 0.$$

Exercise 3. If f is an *odd* function ($f(x) = -f(-x)$ for all x), show that $A_n = 0$.

For $n \geq 1$ we obtain

$$\begin{aligned} B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{1}{n\pi}\left\{ -x\cos(nx)|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\cos(nx)\,dx \right\} \\ &= \frac{(-1)^{n+1}\cdot 2}{n}. \end{aligned}$$

So, if this Fourier series converges, we obtain, for $x \in [-\pi, \pi]$,

$$x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 2}{n} \sin(nx).$$

For our second example let's take $f(x) = x^2$.

Exercise 4. Show that if f is *even* ($f(x) = f(-x)$ for all x), then $B_n = 0$.

Since x^2 is even we only need to compute A_n:

$$A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{2}{3} \pi^2,$$

and for $n \geq 1$

$$\begin{aligned} A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) \, dx \\ &= \frac{1}{n\pi} \left\{ x^2 \sin(nx) \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \sin(nx) \, dx \right\} \\ &= \frac{(-1)^n \cdot 4}{n^2}. \end{aligned}$$

As before, if we do have convergence, then, for $x \in [-\pi, \pi]$,

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 4}{n^2} \cos(nx).$$

6.5. Other Intervals

The first variation on the above is to consider intervals of the form $[-l, l]$. The trick is to rescale the problem: if f is defined for $x \in [-l, l]$, we define g by setting $g(x\pi/l) = f(x)$; then g is defined over $[-\pi, \pi]$, and

$$f(x) = g\left(\frac{x\pi}{l}\right) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right),$$

where A_n and B_n are the coefficients for the function g.

Exercise 5. If a_n and b_n are the coefficients for the function f, show that

$$\begin{aligned} a_n = A_n &= \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) \, dx, \\ b_n = B_n &= \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) \, dx. \end{aligned}$$

The second main variation occurs when we consider half-intervals of type $[0, l]$. Here we can just *extend* f to the whole interval $[-l, l]$, and obtain the Fourier series of the extension. If we do have convergence, then the Fourier series we obtained will be equal to f on the half-interval. Of course, we want to extend f in a sensible way. The two standard ways of doing it are by extending f to be an even function, or an odd function.

Exercise 6. Show that if f is extended to be an odd function, then $A_n = 0$, and

$$B_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) \, dx.$$

Exercise 7. Show that if f is extended to be an even function, then $B_n = 0$, and

$$A_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) \, dx.$$

The reason why you would want to use one type of extension over the other depends on the type of problem at hand. For Dirichlet conditions on the half-interval we want the odd extension; for Neumann conditions we want the even extension.

6.6. Coefficients and Their Decay

In this section we want to view f as being 2π-periodic and defined over the whole line. Going back to the two examples we saw in Section 6.4, when $f(x) = x$ we saw that $|B_n|$ decayed like C/n, for some constant C (in fact, $C = 2$ for that example). Likewise, when $f(x) = x^2$, we had $|A_n|$ decaying like C/n^2. The speed with which coefficients decay is rather important if we are interested in the convergence of the series, and the following theorem is very useful.

Theorem 6.4. *Let f be a 2π-periodic function with $k-1$ continuous derivatives, and whose k-th derivative is piecewise continuous. Then the Fourier coefficients of f decay like C/n^k, where the constant C depends only on f.*

Proof. By induction on k. Let's compute A_n, $n \geq 1$.

$$\begin{aligned}
A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\
&= \frac{1}{n\pi}\left\{ f(x)\sin(nx)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\sin nx\,dx \right\} \\
&= -\frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx.
\end{aligned}$$

By induction this last integral is bounded by some constant over n^{k-1}, so that $|A_n| \leq C/n^k$. Likewise for B_n, $n \geq 1$ we have

$$\begin{aligned}
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\
&= \frac{1}{n\pi}\left\{ -f(x)\cos(nx)|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f'(x)\cos nx\,dx \right\} \\
&= \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx,
\end{aligned}$$

where the last equality is true due to the periodicity of f. (Periodicity is important here. Not only is $f(-\pi) = f(\pi)$, but that is also true for all of f's derivatives.) Again, induction gives us $|B_n| \leq C/n^k$. We are left to do the basis of induction, $k = 1$. That means that f' is piecewise continuous. The same formulas we have above for A_n and B_n are valid, and the easy estimate

$$\left| \int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx \right| \leq \int_{-\pi}^{\pi} |f'(x)|\,dx = C$$

shows that $|A_n|$ decays like C/n. A similar estimate holds for B_n. $\square$

This theorem says that the more derivatives you have in hand, the better the Fourier series will converge. Also true is a converse of this theorem: The faster the coefficients decay, the more derivatives you have for f. We will not prove this fact.

NB: The above theorem *is not optimal*, as shown by the examples we computed. The reader is invited to sharpen the hypotheses of this theorem.

6.7. Linear Algebra to the Rescue: L^2

We now ask ourselves the important question: Why, and how, would (6.1) be true? If we look at our problem from the point of view of Linear Algebra, we are claiming that the function f is in the linear span of the functions 1, $\cos(x)$, $\cos(2x)$, $\ldots$, $\sin(x)$, $\sin(2x)$, etc. So what we have here is a vector space problem – but which vector space?

Definition 6.5. The space $L^2 = L^2([-\pi, \pi])$ is formed by those functions f for which

$$\int_{-\pi}^{\pi} |f(x)|^2\,dx$$

is finite.

Technically the integral in the above definition should be the Lebesgue integral – but we will gloss over this point and work with the Riemann integral here.

Definition 6.6. The L^2 norm of a function f is given (and denoted) by

$$\|f\| = \sqrt{\int_{-\pi}^{\pi} |f(x)|^2\,dx}.$$

The inner product of two functions f and g in L^2 is given by

$$(f, g) = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

The inner product on L^2 plays the role of the dot product on $\mathbb{R}^n$. In fact, if you think of the dot product of two vectors as adding the products of the coordinates, you will see that the inner product does the 'same' for functions (where here x plays the role of vector coordinate).

With this notion of inner product, we see that the orthogonality relations are simply stating that the functions 1, $\cos(x)$, $\cos(2x)$, $\ldots$, $\sin(x)$, $\sin(2x)$, etc, are mutually perpendicular. This is great, since orthogonality implies independence. Since we do have independence, to prove that the trigonometric functions for a basis for L^2, all we need to check is that those functions do span L^2. The next definition establishes our notion of convergence in this vector space.

Definition 6.7. Given $f \in L^2$, denote by $S(f, N)$ the N-th partial sum of the Fourier series of f, namely

$$S(f, N)(x) = \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos(nx) + B_n \sin(nx).$$

We say that $S(f, N)$ converges to f in L^2 if and only if

$$\lim_{N\to\infty} \|f - S(f, N)\| = 0.$$

Convergence in L^2 can be very strange.

Exercise 8. Find a sequence of functions $g_n \in L^2$ such that $\|g_n\| \to 0$, but $g_n(x)$ *does not converge to zero for any value of* x.

Our next result shows that $S(f, N)$ is, in some sense, the best approximation for f. To state the result, let

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx),$$

where the a_n and b_n are any coefficients.

Theorem 6.8. *With g defined as above, we have*

$$\|f - S(f, N)\| \leq \|f - g\|.$$

In other words, of all finite trigonometric sums of order at most N, the one with exactly the Fourier coefficients of f is the one that best approximates f in L^2.

Proof. We will show the following equality:

$$\|f - g\|^2 = \|f - S(f, N)\|^2 + \|g - S(f, N)\|^2. \tag{6.9}$$

As a consequence

$$\|f - g\| \geq \|f - S(f, N)\|,$$

with equality holding if and only if $g = S(f, N)$. In what follows, we will write $S = S(f, N)$. To see that (6.9) is true, we start by expanding the left hand side:

$$\|f-g\|^2 = (f-g, f-g) = (f, f)-2(f, g)+(g, g) = \|f\|^2-2(f, g)+\|g\|^2.$$

Now, add and subtract the quantities $\|S\|^2$ and $2(f, S)$:

$$\|f - g\|^2 = \|f\|^2 - 2(f, S) + \|S\|^2 - \|S\|^2 - 2(f, g) + 2(f, S) + \|g\|^2.$$

Notice that $\|f\|^2 - 2(f, S) + \|S\|^2 = \|f - S\|^2$. The above expression becomes

$$\|f - g\|^2 = \|f - S\|^2 - \|S\|^2 - 2(f, g) + 2(f, S) + \|g\|^2.$$

All that is left to do is see that $\|g - S\|^2 = -\|S\|^2 - 2(f, g) + 2(f, S) + \|g\|^2$. We compute:

$$\begin{aligned} (f, S) &= \frac{A_0}{2}(f, 1) + \sum_{n=1}^{N} A_n(f, \cos(nx)) + B_n(f, \sin(nx)) \\ &= \pi\frac{A_0^2}{2} + \sum_{n=1}^{N} \pi(A_n^2 + B_n^2) = (S, S) = \|S\|^2. \end{aligned}$$

Likewise we find

$$(f, g) = \pi \frac{a_0 A_0}{2} + \sum_{n=1}^{N} \pi(a_0 A_0 + b_0 B_0) = (S, g).$$

The reader is now invited to finish the proof. $\square$

Now take $g = 0$ in (6.9). We get $\|f\|^2 = \|f - S\|^2 + \|S\|^2$, and so $\|f\|^2 \geq \|S\|^2$. Explicitly this reads

$$\frac{A_0^2}{2} + \sum_{n=1}^{N} (A_n^2 + B_n^2) \leq \frac{1}{\pi} \|f\|^2.$$

Since N is arbitrary, we obtain *Bessel's inequality*:

$$\frac{A_0^2}{2} + \sum_{n=1}^{\infty} (A_n^2 + B_n^2) \leq \frac{1}{\pi} \|f\|^2. \tag{6.10}$$

Bessel's inequality is related to Pythagoras' Theorem in $\mathbb{R}^n$. What that theorem says is that the square of the norm of a vector is the sum of the squares of the sizes of each of that vector's components, when projected in orthogonal directions. The reason why we did not get equality in Bessel's inequality is simply because we don't know yet if the orthogonal directions given by the trigonometric functions span the whole of L^2. If we knew that, then Bessel's inequality would become an equality, and vice versa: if we could prove that we have equality in (6.10), then we would know that the trigonometric functions span L^2. We will assume that we have proved that the trigonometric functions span L^2. Then we have:

Theorem 6.11. *(Parseval's identity.) If $f \in L^2$, then*

$$\frac{A_0^2}{2} + \sum_{n=1}^{\infty} (A_n^2 + B_n^2) = \frac{1}{\pi} \|f\|^2.$$

Exercise 9. Apply Parseval's identity to the function $f(x) = x$ and conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercise 10. Apply Parseval's identity to the function $f(x) = x^2$. What can you conclude?

6.8. The Many Faces of Convergence

As we pointed out before, convergence in the space L^2 does not imply convergence for particular values of x.

Definition 6.12. We say that f_n converges pointwise to f if and only if for each $x \in [-\pi, \pi]$ we have $f_n(x) \to f(x)$, as n goes to infinity.

Definition 6.13. We say that f_n converges uniformly to f if and only if for each $\varepsilon > 0$ there is some n_0 depending only on ε (and not on x) such that if $n \geq n_0$, then $|f_n(x) - f(x)| < \varepsilon$.

Roughly speaking, uniform convergence implies not only that for each x we have $f_n(x) \to f(x)$, but that $f_n(x)$ and $f_n(y)$ converge *at the same rate* to $f(x)$ and $f(y)$.

Exercise 11. Show that pointwise convergence implies neither uniform nor L^2 convergence. Show that L^2 convergence implies neither pointwise nor uniform convergence. Show that uniform convergence implies both pointwise and L^2 convergence.

A Fourier series may not converge pointwise: Take $f(x) = x$ and check that $S(f, N)(\pi) = 0$ for all N. We can fix that problem if we redefine f at $-\pi$ and π to be equal to zero, but it seems silly to redefine f so that we can obtain pointwise convergence.

More dramatically, if f is not continuous, then $S(f, N)$ can *never* converge uniformly to f. That is because of a theorem from Analysis that guarantees that the uniform limit of continuous functions must be continuous. Since the $S(f, N)$ are continuous, we can't have uniform convergence to f when f itself is discontinuous.

6.9. The Way Things Are

In this section we give a survey of convergence results for Fourier series. We won't prove any of these results. We assume throughout that $f \in L^2$.

Theorem 6.14. *If f is piecewise differentiable, then $S(f, N)(x)$ converges to $f(x)$ for all values of x for which f is continuous. If x is a point of discontinuity of f, then $S(f, N)(x)$ converges to the average of the lateral limits of f at x.*

Theorem 6.15. *If f is piecewise differentiable and continuous, then $S(f, N)$ converges uniformly to f. The rate of convergence depends on the degree of differentiability of f.*

Theorem 6.16. *(Carleson) If $f \in L^2$ (without smoothness assumptions), then for almost every x we have that $S(f, N)(x)$ converges to $f(x)$.*

(The notion of 'almost every x' has a precise, technical meaning in Analysis. It is fair to say that this theorem is one of the hardest in all of Mathematics.)

Theorem 6.17. *(Gibbs' phenomenon) If f has a discontinuity at x, then $S(f, N)$ has a little bump near x. The height of that bump tends to a constant as N increases, and the bump is always present.*

Of the theorems above the most useful for us are the first two. They were both proved by Dirichlet – indeed, he was the first to show any type of convergence results for Fourier series. (Cauchy tried, but failed.) Here was Dirichlet's approach: If you insert the formula for the A_n and B_n into $S(f, N)$, and exchange the finite sum with the integral, you get

$$\begin{aligned} S(f,N)(x) &= \\ &\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\left(\frac{1}{2} + \sum_{n=1}^{N}(\cos(ny)\cos(nx) + \sin(ny)\sin(nx))\right) dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\left(1 + \sum_{n=1}^{N} 2\cos(n(x-y))\right) dy. \end{aligned}$$

We set

$$D_N(z) = 1 + \sum_{n=1}^{N} 2\cos(nz).$$

This is the so-called Dirichlet kernel, and it can be shown to equal

$$D_N(z) = \frac{\sin(N+\frac{1}{2})z}{\sin\frac{1}{2}z}. \tag{6.18}$$

The function D_N has useful properties that Dirichlet explored to obtain his convergence results.

Exercise 12. Prove formula (6.18). (This could be very hard.)

Exercise 13. What is $\int_{-\pi}^{\pi} D_N(x)\,dx$? (This is very easy!)

Exercise 14. Use Maple to infer a relationship between the height of the bump in Gibbs' phenomenon, and the size of the jump at the discontinuity.