An Introduction to Partial Differential Equations in the Undergraduate Curriculum

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LECTURE 9
Sturm-Liouville Theory—Part II

1.1. Outline of Lecture

- IBVP with nonhomogeneous boundary data
- Sturm-Liouville equations
- Orthogonality
- Eigenvalues and eigenvectors

1.2. The heat equation with nonhomogeneous boundary data

In the previous lecture, we considered the initial boundary value problem (IBVP) with nonhomogeneous boundary data,

$$\begin{aligned} u_t = K u_{xx} &: \quad 0 < x < L, \quad t > 0, \\ \left.\begin{aligned} u(0,t) &= T_1 \\ u_x(L,t) + hu(L,t) &= hT_2 \end{aligned}\right\} &: \quad t > 0, \\ u(x,0) = T_3 &: \quad 0 < x < L. \end{aligned} \tag{1.1}$$

where h, T_1, T_2, T_3, and K were (strictly) positive constants.

We transformed this IBVP into an IBVP with homogeneous boundary data by finding the steady state solution to (1.1) and changing variables to $v(x,t) = u(x,t) - u_s(x)$. The resulting IBVP had the form

$$\begin{aligned} v_t = Kv_{xx} &: \quad 0 < x < L, \quad t > 0 \\ \left.\begin{array}{l} v(0,t) = 0 \\ v_x(L,t) + hv(L,t) = 0 \end{array}\right\} &: \quad t > 0, \\ v(x,0) = T_3 - u_s(x) &: \quad 0 < x < L. \end{aligned}$$

A standard separation of variables argument then led to an ordinary differential equation with boundary conditions: we called this the *Sturm-Liouville problem*. We then showed that this problem had infinitely many orthogonal solutions.

In this lecture, we seek more general conditions under which the Sturm-Liouville ordinary differential equation has infinitely many orthogonal solutions. This study will lead to a description of the **Sturm-Liouville Theory**.

1.3. Sturm-Liouville Equations

Definition 1.2. A **Sturm-Liouville equation** on $x \in [a,b]$ is an ordinary differential equation of the form

$$(s(x)X'(x))' + (\lambda\rho(x) - q(x))\,X(x) = 0 \tag{1.3}$$

where

(a) $s(x), q(x), \rho(x)$ are given continuous functions defined on $[a,b]$ with $s(x)$ and $\rho(x)$ both positive-valued on (a,b), and
(b) λ is an unknown constant called the *eigenvalue parameter*.

The function ρ is called the *weight function* for the Sturm-Liouville equation.

1.3.1. Examples of Sturm-Liouville equations

1. The Sturm-Liouville equation from Lecture 8,

$$X'' = -\lambda X \tag{1.4}$$

satisfies the formal definition (1.2) for $s(x) = 1,\ q(x) = 0$, and weight function $\rho(x) = 1$.
2. Bessel's equation,

$$(x\phi'(x))' + \lambda\,(x\phi(x)) = 0 \tag{1.5}$$

is a Sturm-Liouville equation for $x \in [1,2]$ with $s(x) = x,\ q(x) = 0$, and weight function $\rho(x) = x$.
3. Bessel's equation

$$(x\phi'(x))' + \lambda\,(x\phi(x)) = 0$$

is *not* a Sturm-Liouville equation for $x \in [-1, 1]$ since at $x = 0$ the function s is not positive.

Definition 1.7. In the case that both

$$s(a)\rho(a) \neq 0 \quad \text{and} \quad s(b)\rho(b) \neq 0,$$

the Sturm-Liouville equation (1.3) is called **regular**. If the equation is not regular, it is called **singular**.

1.3.2. Examples of regular and singular Sturm-Liouville equations

1. The eigenfunction equation $X'' = -\lambda X$ is regular since $s(x)\rho(x) = 1 \neq 0$ for all x.
2. Bessel's equation is regular for $x \in [1, 2]$.
3. Bessel's equation is singular for $x \in [0, 1]$.

Sturm-Liouville equations naturally result from separation of variables applied to many IBVP of physical and mathematical interest. Fortunately, these equations with appropriate boundary conditions provide a wealth of orthogonal functions. In fact, we will see that regular Sturm-Liouville problems have an infinite number of eigenvalues, and the corresponding eigenfunctions form a *complete*, orthogonal set.

1.3.3. Solutions of Sturm-Liouville equations

Definition 1.8. A **solution** of the Sturm-Liouville equation (1.3) is defined to be a pair (X, λ) with $X(x)$ a nonzero function and λ a constant. The function X is called the **eigenfunction** and the corresponding λ is called the **eigenvalue**.

1.3.3.1. *Examples.*

1. For the Sturm-Liouville equation $X'' = -\lambda X$ on $[0, \pi]$ with data $X(0) = 0$ and $X(\pi) = 0$, an eigenfunction is given by $X(x) = \sin(nx)$ with corresponding eigenvalue $\lambda = n^2$, for any positive integer n.
2. For the Sturm-Liouville equation $X'' = -\lambda X$ on $[0, L]$ with data $X(0) = 0$ and $X'(L) + hX(L) = 0$, an eigenfunction is $X(x) = \sin\left(x\sqrt{\lambda}\right)$ with corresponding eigenvalue λ given by a positive solution of the transcendental equation $\sqrt{\lambda} + h \tan\left(L\sqrt{\lambda}\right) = 0$.

1.4. Orthogonality

Following Pinsky, we observe that "each Sturm-Liouville equation has its own orthogonality relation, depending on the weight function ρ."

Definition 1.9. Two functions X_1 and X_2 are **orthogonal** with respect to the Sturm-Liouville equation (1.3) if

$$\int_a^b \rho(x) X_1(x) X_2(x) dx = 0. \tag{1.10}$$

This condition is also described as being *orthogonal with respect to the weight function* ρ.

Lemma 1.11. (Pinsky) *Suppose that $\lambda_1 \neq \lambda_2$ are distinct eigenvalues of the Sturm-Liouville equation* (1.3) *with corresponding eigenvectors X_1 and X_2. If the boundary conditions of* (1.3) *satisfy*

$$s(b)\left(X_1(b)X_2'(b) - X_1'(b)X_2(b)\right) = s(a)\left(X_1(a)X_2'(a) - X_1'(a)X_2(a)\right),$$

then X_1 and X_2 are orthogonal with respect to the weight function ρ given in (1.3).

The boundary condition in this lemma may be written more simply as

$$s(x)\left(X_1(x)X_2'(x) - X_1'(x)X_2(x)\right)\Big|_a^b = 0. \tag{1.13}$$

The proof of this lemma is very similar to the energy argument developed in the previous lecture. Let X_1 and X_2 be eigenfunctions corresponding to distinct eigenvalues λ_1 and λ_2. Then,

$$\left(sX_i'\right)' + \left(\lambda_i \rho - q\right) X_i = 0, \quad \text{for } i = 1, 2. \tag{1.14}$$

Multiply equation (1.14) for $i = 1$ by X_2 and integrate the result over $a < x < b$ to find

$$\int_a^b X_2 \left(sX_1'\right)' + \int_a^b X_2 \left(\lambda_1 \rho - q\right) X_1 = 0. \tag{1.15}$$

Integration by parts applied to the first term gives

$$\int_a^b X_2 \left(sX_1'\right)' = X_2 s X_1'\Big|_a^b - \int_a^b s X_1' X_2'. \tag{1.16}$$

Substituting this form into equation (1.15) gives

$$X_2 s X_1'\Big|_a^b - \int_a^b s X_1' X_2' + \int_a^b X_2 \left(\lambda_1 \rho - q\right) X_1 = 0 \tag{1.17}$$

Similarly, multiply equation (1.14) for $i = 2$ by X_1 and integrate the result over $a < x < b$ to find

$$\int_a^b X_1 \left(sX_2'\right)' + \int_a^b X_1 \left(\lambda_2 \rho - q\right) X_2 = 0. \tag{1.18}$$

Substituting this form into equation (1.18) gives

$$X_1 s X_2' \Big|_a^b - \int_a^b sX_2'X_1' + \int_a^b X_1 \left(\lambda_2 \rho - q\right) X_2 = 0 \tag{1.19}$$

Now subtract equation (1.19) from (1.17) and apply the boundary condition (1.13): all terms except for the final integrals will cancel, leaving

$$(\lambda_1 - \lambda_2) \int_a^b \rho X_1 X_2 = 0.$$

Hence, X_1 and X_2 are orthogonal with respect to ρ. This completes the proof.

Notice that the proof of Lemma 1.11 depends on the form of the boundary conditions. It is useful to study these conditions further to determine when the lemma applies.

Definition 1.20. Separable boundary conditions have the form

$$\cos(\alpha)X'(a) - \sin(\alpha)X(a) = 0 \tag{1.21}$$

$$\cos(\beta)X'(b) - \sin(\beta)X(b) = 0 \tag{1.22}$$

where α and β are real constants.

Two functions X_1 and X_2 satisfying boundary conditions (1.21) and (1.22) must necessarily satisfy the boundary condition required by Lemma 1.11, since at both $x = a$ and $x = b$, we may show

$$X_1(x)X_2'(x) - X_1'(x)X_2(x) = 0.$$

Exercise 1. Argue that the above statement is true.

Exercise 2. Show that the following cases are all examples of the separable boundary conditions (1.21) and (1.22).

i. $X(a) = 0$ and $X(b) = 0$ with $\alpha = \pi/2 = \beta$.
ii. $X'(a) = 0$ and $X'(b) = 0$ with $\alpha = 0 = \beta$.
iii. $X'(a) = 0$ and $X'(b) + X(b) = 0$ with $\alpha = 0$ and $\beta = \pi/4$.

Definition 1.23. Periodic boundary conditions have the form

$$X(a) = X(b) \quad \text{and} \quad X'(a) = X'(b)$$

for Sturm-Liouville equations satisfying $s(a) = s(b)$.

1.4.1. Example

For $X'' = -\lambda X$ with periodic boundary conditions $X(0) = X(2\pi)$ and $X'(0) = X'(2\pi)$, the eigenvalues are $\lambda_n = n^2$ with corresponding eigenfunctions $X_n(x) = A_n \cos(nx) + B_n \sin(nx)$ for $n = 0, 1, 2, \ldots$. Notice that $\lambda_0 = 0$ is an eigenvalue since it corresponds to $X_0(x) = A_0$ which is not necessarily zero.

For a given Sturm-Liouville equation, if $s(a) = s(b) = 0$, then the equation is singular. Further, any pair of solutions X_1 and X_2 having continuous derivatives on $[a, b]$ automatically satisfy the boundary condition given in Lemma 1.11. Thus, any pair of such solutions are orthogonal with respect to ρ.

1.5. Eigenvalues and Eigenvectors

1.5.1. Eigenvalues of a regular problem cannot be complex

(Write a proof here.)

1.5.2. Non-negative eigenvalues

Since the eigenvalues are strictly real, we may use an energy argument virtually identical to that in Lecture 8 to demonstrate the following.

Lemma 1.24. *If* (X, λ) *is a solution of the Sturm-Liouville equation* (1.3) *satisfying the boundary condition* (1.13)*, then*

$$\lambda \int_a^b \rho X^2 \geq \int_a^b q X^2 + \int_a^b s (X')^2.$$

If in addition $q(x) \geq 0$*, then all these eigenvalues are non-negative.*

We have seen this in a straightforward argument by cases when given

(a) zero boundary data $X(a) = 0 = X(b)$, or
(b) mixed boundary data $X(a) = 0 = X'(b)$.

1.5.3. Completeness of eigenfunctions

We will discuss completeness of the eigenfunctions only in the case of a regular Sturm-Liouville problem with separable boundary data.

Theorem 1.25. (Pinsky) *There exists an infinite sequence of solutions* (X_j, λ_j) *of the regular Sturm-Liouville problem*

$$(s(x) X'(x))' + (\lambda \rho(x) - q(x)) X(x) = 0$$

with separable boundary conditions

$$\cos(\alpha) X'(a) - \sin(\alpha) X(a) = 0 \quad \text{and} \quad \cos(\beta) X'(b) - \sin(\beta) X(b) = 0.$$

If $f(x)$ *is a smooth*[1] *function on* $[a, b]$ *that satisfies these boundary conditions, then the following series is uniformly convergent on* $[a, b]$*:*

$$\sum_{j=1}^{\infty} A_j X_j(x) = f(x)$$

where the Fourier coefficients are given by the formulas

$$A_j \int_a^b X_j^2(x) \rho(x) dx = \int_a^b f(x) X_j(x) \rho(x) dx \tag{1.27}$$

for $j = 1, 2, \ldots$.

We will not prove this theorem. However, one key idea in its proof is to eliminate the mixed term $(s(x)X'(x))'$ in the Sturm-Liouville equation by a clever change of variables.

Theorem 1.25 is particularly helpful because it allows us to expand an arbitrary smooth function into a *convergent* series of eigenfunctions

[1] By "smooth" we mean continuous with infinitely many continuous derivatives.

of a given Sturm-Liouville problem. This is exactly what we need to compute the Fourier coefficients in an IBVP formal solution so that the initial data will be satisfied.

Notice that this theorem does not prescribe the sign of the eigenvalues of the problem.

1.5.4. Example

Consider the Sturm-Liouville problem on $0 < x < 2$,

$$-X'' = \lambda X$$

$$X(0) + 2X'(0) = 0, \quad 3X(2) + 2X'(2) = 0.$$

Consider the possible eigenvalues, beginning with $\lambda = 0$. In this case $X(x) = Ax + B$. From the boundary conditions, then, $B + 2A = 0$ and $8A + 3B = 0$. The only solution to this system is $A = B = 0$; thus, $\lambda = 0$ is not an eigenvalue. Now let $\lambda = -\omega^2$ so that the corresponding eigenfunction is $X(x) = A\cosh(\omega x) + B\sinh(\omega x)$. The boundary conditions become

$$A + 2\omega B = 0$$

and

$$(3\cosh(2\omega) + 2\omega\sinh(2\omega))A + (3\sinh(2\omega) + 2\omega\cosh(2\omega))B = 0.$$

Appealing to linear algebra, we see that these conditions may be written in matrix form and that the system will have a nonzero solution when the determinant of the coefficient matrix is zero. Hence,

$$\tanh(2\omega) = \frac{4\omega}{3 - 4\omega^2}. \tag{1.28}$$

Exercise 3. Verify that the boundary conditions in this example lead to the stated equations for A and B. Then verify that ω must satisfy this transcendental equation.

Exercise 4. Use Maple to compute the solution of equation (1.28).

The last exercise will demonstrate that there are exactly two nonzero solutions to this equation, $\pm.39$. Hence there is only one negative eigenvalue, $\lambda = -(\pm.39)^2 = -.1521$. Since the theorem guarantees infinitely many eigenvalues, we see that there must exist infinitely many *positive* eigenvalues.

Exercise 5. Find the eigenvalue relation and compute (numerically) the first five eigenvalues.

1.6. Summary

(Following Logan) The theory developed here may be summarized as follows. For a regular Sturm-Liouville problem with separable boundary conditions, there exist infinitely many real eigenvalues. The eigenfunctions corresponding to distinct eigenvalues are orthogonal, and the set of all eigenfunctions is complete in the sense that every square-integrable function f can be expanded in terms of the eigenfunctions.

1.7. Challenge Problems for Lecture 9

Problem 1. (Logan) Show that the Sturm-Liouville problem

$$\begin{aligned} -X''(x) = \lambda X(x) : \qquad & 0 < x < L \\ X'(0) = 0 \qquad \text{and} \qquad & X(L) = 0 \end{aligned}$$

has eigenvalues

$$\lambda_j = \left(\frac{(1+2j)\pi x}{2L}\right)^2$$

with corresponding eigenfunctions

$$X_j(x) = \cos\frac{(1+2j)\pi x}{2L}$$

for $j = 1, 2, \ldots$.

Problem 2. (Logan) Find the eigenvalues and eigenfunctions for the following problem with periodic boundary conditions:

$$\begin{aligned} -X''(x) = \lambda X(x) : \qquad & 0 < x < L \\ X(0) = X(L) \qquad \text{and} \quad & X'(0) = X'(L). \end{aligned}$$

Problem 3. (Logan) Consider the regular Sturm-Liouville problem

$$-X''(x) + q(x)X(x) = \lambda X(x), \quad 0 < x < L,$$

$$X(0) = X(L) = 0$$

where $q(x) > 0$ on $[0, L]$. Show that if λ and X are an eigenvalue and its associated eigenfunction, then

$$\lambda = \frac{\int_0^L \left((X')^2 + qX^2\right) dx}{\|X\|^2}.$$

Is $\lambda > 0$? Can $X(x) = \text{constant}$ be an eigenfunction?

Problem 4. Prove that the eigenvalues are real in the general case

$$(sX')' + (\lambda - q)X = 0$$

with boundary conditions satisfying

$$\begin{aligned} \cos(\alpha)X'(a) - \sin(\alpha)X(a) &= 0, \\ \cos(\beta)X'(b) - \sin(\beta)X(b) &= 0. \end{aligned}$$