PCMI UFP Program
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PDE's & Maple Lab 1

1 Easy

You can download a MAPLE worksheet that does these problems from:

`http://www.math.hmc.edu/~ajb/PCMI/PDE_Lab1.mws`

1. Consider the convection equation

$$u_t + cu_x = 0, \qquad x \in \mathbb{R}, t > 0, \tag{1}$$

and let $F(x) = e^{-x^2}$.

   (a) Show $u(x, t) = F(x - ct)$ solves (1).
   (b) Let $c = 1$. Plot $u(x, t)$ at time $t = 0, 1, 2$ on the same axes.
   (c) Plot the solution surface $u(x, t)$ for $-6 < x < 6$ and $0 \leq t < 2$.
   (d) Create an animation of the solution $F(x - t)$ for $0 < t < 2$.

2. Consider Laplace's equation in $\mathbb{R}^2$:

$$\Phi_{xx} + \Phi_{yy} = 0, \qquad (x, y) \in \mathbb{R}^2, \tag{2}$$

and let $F(x, y) = e^{-x} \cos y$.

   (a) Show F solves (2).
   (b) Plot $F(x, y)$ for $0 < x < 1$ and $0 < y < 2\pi$.

3. Consider the function $f(x) = x$ on the interval $[0, \pi]$. The Fourier sine series for f is given by

$$f(x) = 2 \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin mx. \tag{3}$$

Plot the function f together with its Fourier approximation taking 2, 4, 8, 16 and then 32 terms of the series.

2 Medium

1. Consider the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad x \in \mathbb{R},\ t > 0, \tag{4}$$

and let $F(x) = e^{-x^2}$.

   (a) Show $u(x,t) = \frac{1}{2}\left(F(x-ct) + F(x+ct)\right)$ solves (4).

   (b) Let $c = 1$. Plot $u(x,t)$ at time $t = 0, 1, 2$ on the same axes.

   (c) Plot the solution surface $u(x,t)$ for $-6 < x < 6$ and $0 \leq t < 2$.

   (d) Create an animation of the solution $u(x,t)$ for $0 < t < 2$.

2. Consider the heat equation

$$u_t = u_{xx}, \qquad x \in \mathbb{R},\ t > 0, \tag{5}$$

and let $u(x,t) = \frac{1}{\sqrt{4\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}}$.

   (a) Show $u(x,t)$ solves (5).

   (b) Plot $u(x,t)$ at time $t = 0, 1, 2$ on the same axes.

   (c) Plot the solution surface $u(x,t)$ for $-6 < x < 6$ and $0 \leq t < 2$.

   (d) Create an animation of the solution $u(x,t)$ for $0 < t < 2$.

   (e) Show that $\int_{-\infty}^{\infty} u(x,t)\, dx = 1$ for each $t > 0$.

3. Consider the following initial boundary value problem for the heat equation:

$$\begin{cases} u_t = u_{xx}, & x \in (0,\pi),\ t > 0, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = x, & x \in [0,\pi]. \end{cases} \tag{6}$$

The Fourier series solution to this is given by

$$u(x,t) = \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{m} e^{-m^2 t} \sin mx.$$

Plot the 16 term approximation to the solution $u(x,t)$ at time $t = 0, 1, 2$. Create an animation of the solution $u(x,t)$ for $0 < t < 4$.

3 Challenge

1. **Harmonic Polynomials.**

   (a) Show $F(x, y) = x^3 - 3xy^2$ is harmonic on $\mathbb{R}^2$. Plot F.

   (b) Find all cubic harmonic polynomials, i.e., all harmonic polynomials of the form $H(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$.

2. **d'Alembert's Solution.** Let $u(x, t)$ be defined by

$$u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} e^{-s^2} \, ds. \tag{7}$$

   (a) Show $u(x, t)$ solves the wave equation $u_{tt} = u_{xx}$. What is the initial displacement and velocity?

   (b) Plot $u(x, t)$ at time $t = 0, 1, 2$ on the same axes.

   (c) Plot the solution surface $u(x, t)$ for $-6 < x < 6$ and $0 \leq t < 2$.

   (d) Create an animation of the solution $u(x, t)$ for $0 < t < 2$.

3. **The Erf Function.** Let $v(x)$ be defined by

$$v(x) = \int_{0}^{x} e^{-s^2} \, ds. \tag{8}$$

   (a) Show $u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} v\left(\frac{x}{\sqrt{4t}}\right)$ solves the heat equation $u_t = u_{xx}$.

   (b) Plot $u(x, t)$ at time $t = .5, 1, 2$ on the same axes. What is the initial temperature distribution?

   (c) Plot the solution surface $u(x, t)$ for $-6 < x < 6$ and $0 \leq t < 2$.

   (d) Create an animation of the solution $u(x, t)$ for $0 < t < 2$.

   (e) Show $u_x(x, t)$ also solves the heat equation. Find and plot this solution. What is its initial temperature distribution?