PCMI-UFP
Monday, July 14, 2003

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Problem Solving Seminar # 2

Parity and Invariants

This problem session is modelled after the Harvey Mudd College Putnam Problem Solving Seminar, which runs every Tuesday night in the fall semester in preparation for the annual Putnam Mathematics Competition.

B1: (a) Place a knight on each square of a 7×7 chessboard. Is it possible for each knight to simultaneously make a legal move so that each knight ends up in its own square? (Larson)

(b) Suppose we have a knight on a 4×4 chessboard. Can it make a sequence of legal moves visiting each square exactly once?

B2: (a) Show that for any set of three integers, we can find two of them whose average is also an integer.

(b) Suppose we have a set of integers with the property that the average of any three of them is not an integer. Determine the maximum number of elements the set can have.

B3: Given a 5×5 chessboard with one square removed, we wish to tile the rest of the board with 1×3 blocks. For which removed squares is this possible, and for which is it not?

B4: On a tropical island live R red chameleons, B blue chameleons and Y yellow chameleons. When two chameleons of different color meet they change to the third color. For what triples (R, B, Y) can all the chameleons eventually be the same color?

B5: A deck of 52 playing cards is printed on a large rectangular sheet consisting of four rows of thirteen cards each. What is the minimum number of linear cuts needed to separate the cards, assuming that piling (but not folding) is permitted? Is there a more efficient arrangement of the cards on a rectangular sheet that reduces the number of cuts?

And now for something completely different . . .

B6: Not quite origami . . . Cut the central square out of a 5×5 grid of 25 squares. Can you make a single cut through the resulting shape so that the two pieces can be arranged, by various folds, into the surface of a $2 \times 2 \times 2$ cube? (A "cut" is a piecewise linear cut.)

(Quantum Magazine)

Hints:

1. a) Black and white squares may be useful here. b) Suppose we color the center four squares black and the corners white. What do we know about a move to/from a white square.

2. a) Consider the remainders for each number when dividing by two. b) Now consider the remainders when dividing by three.

3. Can you color the board with three colors in a way that is helpful?

4. Start with three chameleons and work your way up.

5. For every cut, each card is on the right or the left side. What does this tell you?