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**Problem Solving Seminar # 1**

# The pigeonhole principle

This problem session is modelled after the Harvey Mudd College Putnam Problem Solving Seminar, which runs every Tuesday night in the fall semester in preparation for the annual Putnam Mathematics Competition.

**A1:** (a) Show that among any 6 points in a $3 \times 4$ rectangle there is a pair of points not more than $\sqrt{5}$ apart.

(b) Show that among any 9 points in a triangle of area 1, there are 3 points that form a triangle of area at most 1/4. (Boltyanski & Soifer)

(c) Show that given any 9 points in a triangle of area 1, there is a triangle of area at least 1/12 that does not contain any of those 9 points in its interior. (Can you improve 1/12?)

**A2:** Show that any odd number not divisible by 5 must divide some number of the form 10101...01, an alternating string of 1's and 0's.

For example, 13 divides 10101, 17 divides 101010101010101, 9 and 19 divide 10101010101010101.

**A3:** Given any set $A$ of ten integers between 1 and 99, show that there are two disjoint non-empty subsets of $A$ with equal sums of their elements. (Larson)

**A4:** (a) Show that there is some four-digit combination that occurs infinitely often as the first four digits of the powers of 2.

(b) Show that 2004 occurs infinitely often as the first four digits of the powers of 2.

**A5:** Given any 5 distinct points on the surface of a sphere, show there exists a closed hemisphere that contains at least 4 of them. (Putnam 2002)

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And now for something completely different . . .

**A6: [Yet another hat problem.]** I'm going to give each of you a hat to wear that is either black or white. You cannot see a hat on your head, but you can see a hat on anyone else's head. Your Goal as a group is to organize yourselves in a line, with all the white-hatted folks on one end, and all the black-hatted folks on the other end. Note that after I give you your hats, you cannot communicate with (nor touch) anyone. However, before I give you the hats you may jointly decide on a strategy.

(a) Devise a strategy that will achieve the Goal.

(b) Can you find a symmetric strategy, i.e., one that is the same for all players?

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Hints:

1. (a) What are the pigeonholes? (b) Start with an equilateral triangle.
2. As a warm-up, show that among any 5 numbers there are two numbers whose difference is a multiple of 4.
3. How many subsets are there of size 10? What's the largest possible sum of 10 numbers?
4. (b) The logarithm (base 10) of 2 is irrational...
5. What's the fewest number of hemispheres that will cover a sphere (perhaps with overlap)?