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Pizza & Problem Solving # 2

Cubes and other geometric objects

This problem session is modelled after the Harvey Mudd College Putnam Problem Solving Seminar, which runs every Tuesday night in the fall semester in preparation for the annual Putnam Mathematics Competition.

Want to run a problem-solving group at your school? Come to the UFP talk: "Running a Problem Solving Group & Other Ways to Encourage Math Community", 1pm tomorrow, Coalition 4

B1: (a) Suppose you are given a $3 \times 3 \times 3$ cube of cheese, and a large flat knife. Try to cut the cube into $1 \times 1 \times 1$ cubes using as few cuts as possible. (You may move pieces around between cuts so as to maximize the benefit of each cut.) What's the fewest number of cuts you need?

(b) What's the fewest cuts needed for a $4 \times 4 \times 4$ cube (to be cut into $1 \times 1 \times 1$ cubes)?

(c) What about a $5 \times 5 \times 5$ cube?

B2: Suppose that a regular octagon is tiled with non-overlapping parallelograms. Show that at least 2 of these parallelograms are rectangles.

B3: Must every (bounded, convex) polyhedron* have two faces with the same number of edges? (*For this crowd: a 3-dim'l polytope.) (Barbeau, Klamkin, Moser)

B4: A cubical chessboard? Consider an $8 \times 8 \times 8$ cube with two $1 \times 1 \times 1$ cubes removed from diagonally opposite corners. For which n is it possible to completely fill this object with $1 \times 1 \times n$ boxes? (Wagon)

B5: Tropical cubes. In *tropical* arithmetic on real numbers, $a \otimes b = a + b$, and $a \oplus b = \min(a, b)$. We can also do tropical operations on $n \times n$ matrices! The sum of matrices A, B is obtained by taking the minimums of the corresponding entries. The product of A and B has its ij-th entry still given by the dot product of row i of A with column j of B, but $\times$ and $+$ in that dot product are interpreted tropically.

(a) Let A be a 2×2 matrix, not all zeroes. Find an infinite class of examples where $A^3 = A$ tropically, but $A^2 \neq A$.

(b) Let B be a 3×3 matrix, not all zeroes. If $B^2 \neq B$, is it possible for $B^3 = B$ tropically?

And for a little bit of variety...

B6. Shoelace Clock. You are given some matches, a shoelace, and a pair of scissors. The shoelace burns like a fuse at a non-constant rate and takes 60 minutes to burn. However, it has a symmetry property: the burn rate at a distance x from the left end is the same as the burn rate at the same distance x from the right end. What is the smallest time interval that you can accurately measure with this shoelace?

Hints:

B1.(a) Clearly 6 cuts are enough, if you don't move pieces around. Can you do better?

B2. Look for structure: consider a path of parallelograms from one side to the opposite side.

B3. Suppose there are n faces... and how many pigeons?

B4. In R. Stanley's Clay Lecture, he showed that the (planar) chessboard with opposite corners removed can't be covered by dominoes, because...?

B5. In B. Sturmfels's Clay Lecture, we saw that tropical mathematics can have some strange properties!

This problem set (and other problem-solving resources) will be posted at the PCMI Undergraduate Faculty Program site: `http://www.math.hmc.edu/~su/pcmi/`.