PCMI
Thursday, June 30, 2005

Andrew J. Bernoff
Harvey Mudd College

Pizza & Problem Solving # 1

The Sum of the Parts

Some PCMI problem solving resources, including a copy of this problem set can be found at:

`http://www.math.hmc.edu/~ajb/PCMI/problem_solve.html`

A1: (a) Prove that a triangle with sides of length $(5, 5, 6)$ has the same area as a triangle with sides of length $(5, 5, 8)$.

(b) Find an infinite set of other pairs of incongruent isoceles triangles with integer sides having equal area.

(c) Do there exist incongruent isoceles triangles with integer sides having the same area and having no sides of the same length?

A2: Show that you can cut any triangle into three pieces that can be rotated and translated (but not flipped) into a mirror image of the original triangle. (Wagon)

A3: Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 and the remaining four sides of length 2. (Putnam, 1978)

A4: In how many ways can a positive integer n be written as a sum of two or more positive integers if sums with different orders are considered distinct. For example, for $n = 3$, there are three ways, namely

$$3 = 1 + 1 + 1 = 1 + 2 = 2 + 1.$$

A5: (a) Show how to cut a 9×16 rectangle into two pieces that can be assembled into a 12×12 square.

(b) Show how to cut an $8 \times 8 \times 27$ rectangular solid into four pieces that can be assembled into a cube of side 12. (The Moscow Puzzles)

And for a little bit of variety . . .

A6: Suppose you have a polygon of perimeter 12, whose vertices are all on lattice points, whose sides all have integer lengths, and whose area, A, is an integer. Show that A can be $3, 4, 5, 6, 7, 8$ or 9. (Inspired by Wagon)

Hints:

1. (a) A two piece dissection does the trick. (b) Can you generalize the idea in part (a)?
2. Can you break the triangle into three pieces with mirror symmetry?
3. Suppose you have an octagonal pizza; can you rearrange the slices to make the problem easier?
4. Draw a number line . . . how can you visualize a sum on it?
5. (a) Try drawing a 9×16 grid on the rectangle. What's next?
6. Start with rectangles and triangles with integer length sides.