PCMI
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Pizza & Problem Solving # 2

The Extreme Principle

This problem session is modelled after the Harvey Mudd College Putnam Problem Solving Seminar, which runs every Tuesday night in the fall semester in preparation for the annual Putnam Mathematics Competition.

B1: Eight people sit around a lunch table at PCMI. As it happens, each person's age is the average of the two persons' ages on his/her right and left. Show that all their ages are equal.

B2: Fifteen sheets of paper of various sizes and shapes lie on a desktop covering it completely. The sheets may overlap and may even hang over the edge. Show that five of the sheets may be removed so that the remaining ten sheets cover at least 2/3 of the desktop. (Wagon)

B3: Place the integers $1, 2, 3, ..., n^2$ (without duplication) in any order onto an $n \times n$ chessboard, with one integer per square. Show that there exist two (horizontally, vertically, or diagonally) adjacent squares whose values differ by at least $n + 1$. (Zeitz)

B4: Let $p(x)$ be a real polynomial such that for all x, $p(x) + p'(x) \geq 0$. Does it follow that for all x, $p(x) \geq 0$? [For example, $p(x) = x^2 + 1$ satisfies the condition and conclusion.]

B5: Consider finitely many points in the plane such that, if we choose any three points A, B, C among them, the area of triangle ABC is always less than 1. Show that all these points lie within the interior or on the boundary of a triangle of area less than 4. (Korea, 1995)

B6: Let A be a set of $2n$ points in the plane, no three of which are collinear. Suppose that n of them are colored red, and the remaining n blue. Prove or disprove: there are n straight line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors. (Putnam, 1979)

And for a little bit of variety . . .

B7: Strange biology. The DNA of alien creatures on Planet Alpha Lyra IV is very strange; it may be represented as strings of two letters (nucleotides) A and B, it never includes 3 consecutive repetitions of any sequence (of any length) nor does the repetition BB ever occur. What is the longest Lyran DNA sequence? (Hess, variant)

Hints:

1. Who's the youngest?
2. Which piece of paper would you remove first?
3. Any two squares are separated by a path of adjacent squares that is how long?
4. If a polynomial is non-negative, what can be said about its degree? Does $p(x)$ achieve a minimum somewhere?
5. Which triangle should you focus on?
6. Consider the solution that minimizes what quantity?