PCMI
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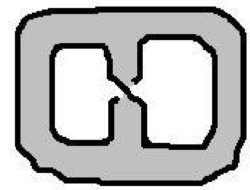
Pizza & Problem Solving # 1
Paths and Curves and Cycles

This problem session is modelled after the Harvey Mudd College Putnam Problem Solving Seminar, co-led with Andrew J. Bernoff. Our problem-solving resources webpage, which includes this problem set and all previous PCMI problem sets, can be found by Googling "PCMI Problem Solving Resources".

A1: The figure shows a piece of paper that by cutting, twisting, and rejoining, has been given one side, two edges, and two "holes". Can you make a similar paper figure that has two sides, two edges, and five "holes"? (Barr, variant)



A2: You are given two potatoes of different shapes and sizes. You are allowed to draw one closed path on the surface of each. For any pair of potatoes, it always possible to draw the two paths so that they are geometrically congruent curves (*isometric* curves) in 3-dimensional space? (You may assume the potatoes are *premium potatoes*, i.e., sufficiently smooth.) (Hess)

A3: A *graph* is a collection of vertices (dots) connected by edges, and a *tree* is a graph that has no *cycles* (closed paths) in it. A *leaf* of a tree is a vertex that is the endpoint of exactly one edge. Show that a tree with two or more vertices has at least two leaves. (Zeitz)

A4: There are 7 ladders each having a banana at the top and a monkey at the bottom. A number of ropes connect the rungs of two different ladders such that no rung has more than one rope tied to it. The monkeys climb up the ladders; each time they encounter a rope, they climb along it to the other end and continue climbing upwards. Show that each monkey gets a banana. (Hess)

A5: (a) A king has four prisoners, numbered 1, 2, 3, 4, who each have a hat with his own number on it. In a vault there are four boxes numbered 1, 2, 3, 4. The king takes all the prisoners' hats and places them *randomly* in the boxes, one to each box. One by one, each prisoner will be allowed to enter the vault and open any 2 of the 4 boxes to try to find his own hat; then he must close the boxes and exit the vault, leaving it just as it was when he entered. If every prisoner finds his own hat, then the king will let all the prisoners live; otherwise all will be hanged.

The prisoners can jointly decide on a strategy before any of them enter the vault, but once they begin they can no longer communicate with each other. Find a strategy that will give the prisoners better than 1/16 chance of survival.

(b) Generalize your strategy for part (a) to the case of 2N prisoners who may sequentially enter the vault and look in N boxes. (Milterson-Gál)

And for a little bit of variety . . .

A6: Are you Positive about this Polynomial? Suppose $f(x)$ is an unknown polynomial with unknown degree and non-negative integer coefficients. Your goal is to determine $f(x)$, but you are only allowed to ask questions of the following form: for a specific number k, "what is $f(k)$?". What is the fewest number of questions needed to determine $f(x)$? (Keene)

Hints:

2. There's an elegant solution by looking at this the right way.
3. Seems easy but be sure your argument is rigorous. The extreme principle may help.
4. Think about the squares between the rungs as part of a monkey path. Which ones are "adjacent" in a monkey path?
5. Follow a path of boxes...
6. Can you find an upper bound for the largest coefficient?