PCMI | **Pizza & Problem Solving # 1** | Andrew J. Bernoff
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The Pigeonhole Principle

This problem session is modelled after the HMC Putnam Preparation Problem Solving Seminar co-led with Francis Su. Additional resources (and this problem set) can be found at:

`http://www.math.hmc.edu/~ajb/PCMI/problem_solve.html`

A1: Show that every convex polyhedron has two faces with the same number of edges. (Barbeau, Klamkin & Moser)

A2: Show that in any finite gathering of people, there are at least two people who know the same number of people at the gathering (assume that "knowing" is a mutual relationship). (Zeitz)

A3:(a) Consider any seven points in a 3×4 rectangle. Prove that some pair of points must be separated by a distance less than or equal to $\sqrt{5}$.

(b) Now consider any **six** points in a 3×4 rectangle. Prove that some pair of points must be separated by a distance less than or equal to $\sqrt{5}$.

A4: On a 3×7 chequerboard, every square is colored red or blue. Show that in any such coloring, there is a rectangle (formed by the lines of the board) whose distinct corner squares are all the same color. (Larson)

A5: Show that if $(n+1)$ numbers are chosen from $\{1, 2, 3, \cdots 2n\}$ one of them is divisible by another. (Engel)

A6: (a) Let $\|x\|$ denote the distance of x to the nearest integer. Show that for any integer m there is some integer $1 \leq n \leq m$ such that $\|n\sqrt{2}\| < 1/m$.

(b) Show there are an infinite number of rational numbers p/q such that $|\sqrt{2} - p/q| < 1/q^2$.

And for a little bit of variety...

A7: The professors in the Undergraduate Faculty Program would like to know their average salary. However, they are self-conscious and don't want to tell each other their own salaries. They are hanging out in a conference room with a simple calculator and not much else. Devise a strategy for them to determine their average salary, without disclosing their own salaries? (adapted from Wu)

Hints:

1. What is the maximum number of edges a face can have? What is the minimum?
2. There is one special case in this problem; can you find it?
3. (a) Can you dissect the rectangle into six smaller rectangles?
4. How many different types of 3×1 columns are there? Are some of them special?
5. First, can you find a set of numbers such that for any pair chosen from the set, one of them is divisible by the other.
6. (a) Choosing pigeonholes carefully here gets you a solution. (b) Part (a) finds you a first solution; how do you find the next one?