Parity & Counting

This problem session is modelled after the HMC Putnam Preparation Problem Solving Seminar co-led with Francis Su. Additional resources (and this problem set) can be found at:

`http://www.math.hmc.edu/~ajb/PCMI/problem_solve.html`

B1: A house has one entrance and many rooms. Every room has 1, 2 or 4 doors, and these doors lead directly to other rooms or to the outside. The rooms with 1 door are precisely the bathrooms. Prove that this house must have an odd number of bathrooms.

B2: (a) Place a knight on each square of a 7×7 chessboard. Is it possible for each knight to simultaneously make a legal move so that each knight ends up in its own square? (Larson)

(b) Place a knight on each square of a 4×4 chessboard. Is it possible for each knight to simultaneously make a legal move so that each knight ends up in its own square?

(c) Suppose we have a knight on a 4×4 chessboard. Can it make a sequence of legal moves visiting each square exactly once?

B3: (a) A deck of 52 playing cards is printed on a large rectangular sheet consisting of four rows of thirteen cards each. What is the minimum number of linear cuts needed to separate the cards, assuming that piling (but not folding) is permitted? Is there a more efficient arrangement of the cards on a rectangular sheet that reduces the number of cuts?

(b) A corn farmer has a set of four weights that each weigh an integer number of pounds and a fair balance. She claims that she can measure out any integer number of pounds of corn up to a maximum of N using just these weights. What is the maximum value of N and what should the weights be?

B4: (a) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$? For example, with $n = 3$ there are four ways: 1+1+1, 1+2, 2+1, 3.

(b) Suppose we now specify that $a_1 \leq a_2 \leq \cdots a_k \leq a_1 + 1$, how many ways can we write n as a sum of positive integers now? For example, with $n = 4$ there are four ways: 1+1+1+1, 1+1+2, 2+2, 4. (Putnam 2003)

And for a little bit of variety...

B5: A deck of 52 playing cards is sitting on a table in a dark room. You are told that all the spades are face up and that all the remaining cards are face down. Your challenge is to arrange the cards into two piles each with the same number of cards face up without turning the lights on. How do you do it? (A former Car Talk Puzzler)

Hints:

1. Can you write a formula relating the number of doors and rooms of various types?
2. (a,b) Think black and white. (c) Can you divide the squares into useful groups?
3. (a) For every cut, each card is on either one side of the cut or the other. For all the cards to be separated, what has to happen with these cuts?
   (b) Can you count the number of arrangements of the weights?
4. (a) Suppose you have a chocolate bar which his $1 \times n$. How many ways can you break it into pieces. (b) Work out a few example and use induction.
5. There are at least two solutions.