# The Extreme Principle

This problem session is modelled after the HMC Putnam Preparation Problem Solving Seminar. Additional resources (and this problem set) can be found at:

`http://www.math.hmc.edu/~ajb/PCMI/problem_solve.html`

**B1:** (a) Eight people sit around a lunch table under the PCMI Tent. As it happens, each person's age is the average of the two persons' ages on his/her right and left. Show that all their ages are equal.

(b) Zack and his ten PCMI buddies sit around a dinner table at the Wasatch Brewery. As it happens, each person's age differs from the two persons' ages on his/her right and left by at most one year. Zack is 26 years old. Can he order a pitcher of beer to share?

**B2:** Fifteen sheets of paper of various sizes and shapes lie on a desktop covering it completely. The sheets may overlap and may even hang over the edge. Show that five of the sheets may be removed so that the remaining ten sheets cover at least 2/3 of the desktop.

**B3:** Place the integers $1, 2, ..., n^2$ (without duplication) in any order onto an $n \times n$ chessboard, with one integer per square. Show that there exist two (horizontally, vertically, or diagonally) adjacent squares whose values differ by at least $n + 1$. (Zeitz)

**B4:** (a) Let $p(x)$ be a polynomial such that for all $x$, $p(x) + p'(x) \geq 0$. Does it follow that for all $x$, $p(x) \geq 0$? [For example $p(x) = x^2 + 1$ satisfies the condition and conclusion.]

(b) Suppose $p(x)$ is a smooth function, but not necessarily a polynomial. Does the answer to (a) change?

**B5:** Consider finitely many points in the plane such that, if we choose any three points $A, B, C$ among them, the area of triangle $ABC$ is always less than 1. Show that all these points lie within the interior or on the boundary of a triangle of area less than 4. (Korea, 1995)

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And for a little bit of variety...

**B6: Fair Division** A father bakes brownies in a rectangular pan as an after school snack for her two daughters. He bakes the brownies and and lets them cool. Before his daughters come home, his wife comes along and cuts some rectangle out of the middle, at random, with the sides not necessarily parallel to the sides of the pan. How can he make one straight cut and divide the remainder of the brownies evenly between his two daughters, so that they get the same area? (Car Talk Puzzler)

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Hints:

1. Think about the oldest person at the table.
2. Which pieces of paper would you remove first?
3. Any two squares are separated by a path of adjacent squares that is how long?
4. (a) If a polynomial is non-negative, what can be said about its degree? Does $p(x)$ achieve a minimum somewhere?
5. Which triangle should you focus on?