Digits & Divisibility

This problem session is modelled after the HMC Putnam Preparation Problem Solving Seminar. Additional resources (and this problem set) can be found at:

`http://www.math.hmc.edu/~ajb/PCMI/problem_solve.html`

B1: What arrangement of the digit $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ into the product of five two-digit numbers yields the largest product? For example, $10 \times 23 \times 45 \times 67 \times 89 = 61,717,050$.
(Kornhauser, Velleman & Wagon)

B2: Find a nine-digit number made up of $1, 2, 3, 4, 5, 6, 7, 8, 9$ in some order such that when digits are removed one at a time from the right the number remaining is divisible in turn by $8, 7, 6, 5, 4, 3, 2, 1$.
(Hess)

B3: Find all 4-digit perfect squares of the form $aabb$. (Engel)

B4: Find all 4-digit numbers that satisfy $4 \cdot abcd = dcba$. (Engel)

B5: Consider the equalities:

$$11 = 2 + (3)^2 \quad 1111 = 22 + (33)^2 \quad 111111 = 222 + (333)^2$$

Make a conjecture about the pattern and prove it. (Andreescu & Gelca)

B6: How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1? (Putnam, 1989)

And for a little bit of variety...

B7: Multiplicative Magic Squares: A Multiplicative Magic Square (MMS) is an $N \times N$ square array of positive integers with the property that the product of the entries in any row, column or along the main diagonals is the same integer, called the product number.

a) Show for any four integers A, B, C, D there exists a 4×4 MMS with entries chosen from A, B, C, D and whose product number is $ABCD$.

b) Find a 4×4 MMS whose entries are all distinct powers of two.

c) (Hard!) Find a 4×4 MMS whose entries are distinct positive integers and whose product number is 5040.
(Sayles)

Hints:

1. Can you solve the analogous problem for two two-digit numbers? What can you say about the arrangement which maximizes the product?
2. Trial and error and divisibility rules go a long way here.
3. Can you identify one factor of the number?
4. Can you determine what a is first?
5. How can you write the number $111 \cdots 1$ with n digits?
6. Does multiplying the number by 99 help?
7. a) You might start out with $A = B = C = 1$. b) How is this related to an additive magic square? c) Can you figure out how to multiply two MMS's to produce a new MMS?