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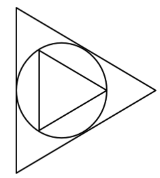
Pizza & Problem Solving

Symmetry

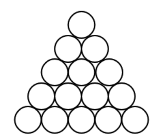
This problem session is modelled after the Harvey Mudd College Putnam Problem Solving Seminar, co-led with Andrew Bernoff. Our problem-solving resources webpage, which includes many previous PCMI problem sets, can be found by Googling "PCMI Problem Solving Resources".



A1: An equilateral triangle is inscribed in a circle which is inscribed in an equilateral triangle. Without pen and paper, find the ratio of the areas of the two triangles.

A2: Consider this 2-player game. Each player takes turns placing a penny on a rectangular table. No penny can touch a penny that is already on the table. The table starts out with no pennies. The last player who makes a legal move wins. Does the first player have a winning strategy? (Zeitz)

A3: Catherine drives a car with a 6-digit odometer. One day while driving, she notes that the last 4 digits on the odometer are palindromic (the same read forwards as backwards). A mile later, the last 5 digits are palindromic. A mile after that, the middle 4 digits are palindromic. And a mile after that, all 6 are palindromic! What was the odometer reading when Catherine first looked at it? (Car Talk Puzzler)



A4: a) Fifteen oranges, packed in a triangular crate, are either smooth or rough-skinned. Show that there are three oranges of the same skin-type whose centers are vertices of an equilateral triangle. (Larson)
b) What's the smallest triangular crate with the above property?

c) Sixteen apples, packed in a square crate in grid-like fashion, are either red or green. Are there four apples of the same color whose centers form the vertices of a square? A challenge: What's the smallest square crate with this property?

A5: With a compass draw a circle on a plane. Without changing the angle of the compass, draw a second circle on a suitably large sphere. Which is larger, the area of the plane enclosed by the first circle, or the area of the sphere's surface enclosed by the second circle? (Hess)

A6: Let n be an odd integer greater than 1. Let A be an n by n symmetric matrix such that each row and each column of A consists of some permutation of the integers $1, ..., n$. Show that each one of the integers $1, ..., n$ must appear in the main diagonal of A. (Putnam 1954)

And for a little bit of variety . . .

A7: a) Ten prisoners are in a bad situation. Tomorrow, the prison warden will randomly arrange them in a line, facing towards one end of the line, and put a red or blue hat on each. Each will see the hats of those in front, but not of those behind or on his own head. One by one, starting at the back of the line and moving forward, the warden will ask each the color of his own hat. Each prisoner must answer, and if he will live if and only if he answers correctly.

Tomorrow the prisoners may not talk to each other, though they will hear the answers to the warden's questions. But tonight they can discuss and devise a strategy to maximize the number of people who will live tomorrow. What is the best strategy and how many will it save?

b) Now suppose there are N colors for the hats and answer the same question.

Hints:

4. If there aren't 3 oranges that form an equilateral triangle, see if you can obtain a contradiction. Exploit symmetry by making arguments "without loss of generality".
5. Calculus can show that in a sphere of radius R, the area of a spherical cap of height h is $2\pi Rh$.
6. What symmetry can be exploited?
7. What can be communicated in a single answer?