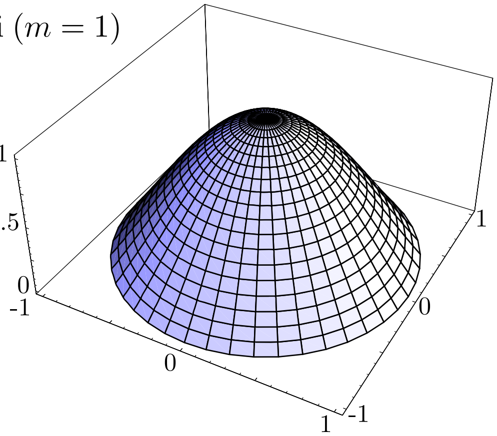
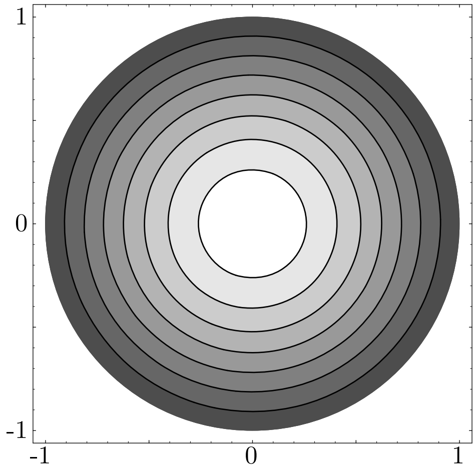
Vibrational modes of a timpani ($m = 1$)

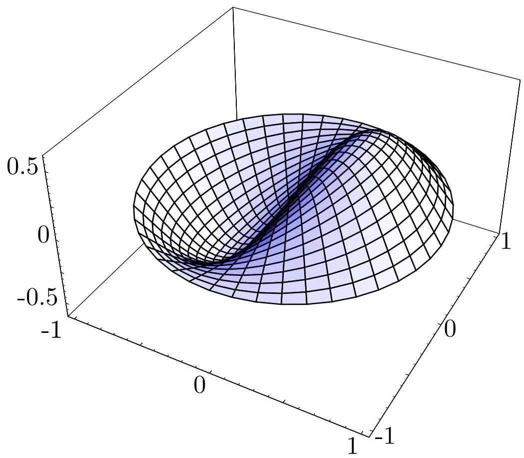
$$J_n\left(\frac{z_{mn}r}{a}\right)\cos(n\theta)\cos\left(\frac{cz_{mn}t}{a}\right)$$

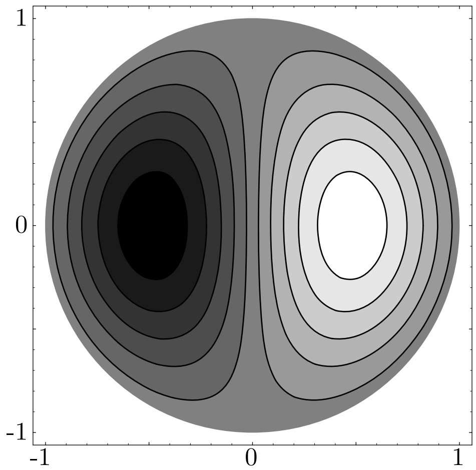
$(m, n) = (1, 0)$



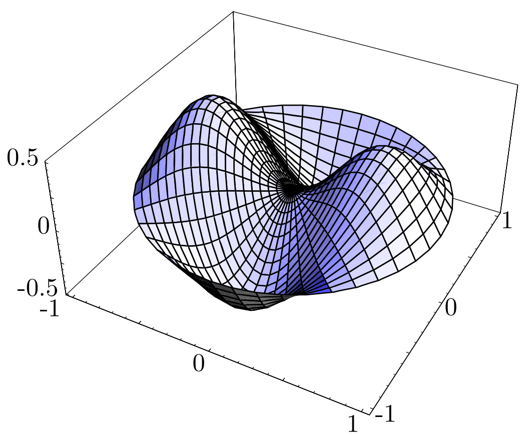


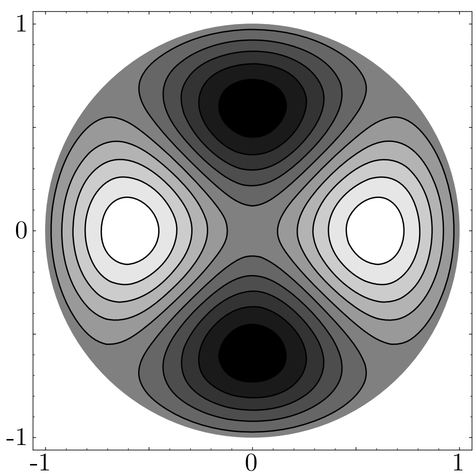
$(m, n) = (1, 1)$



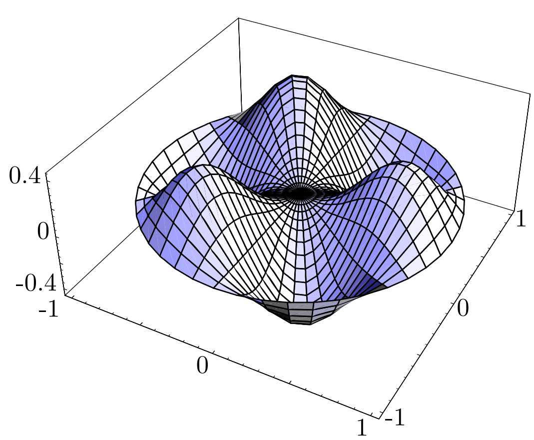


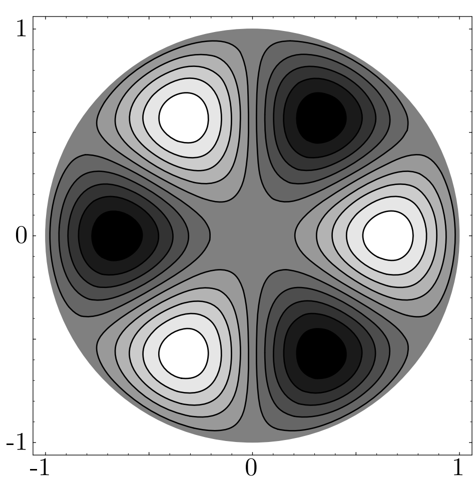
$(m, n) = (1, 2)$





$(m, n) = (1, 3)$

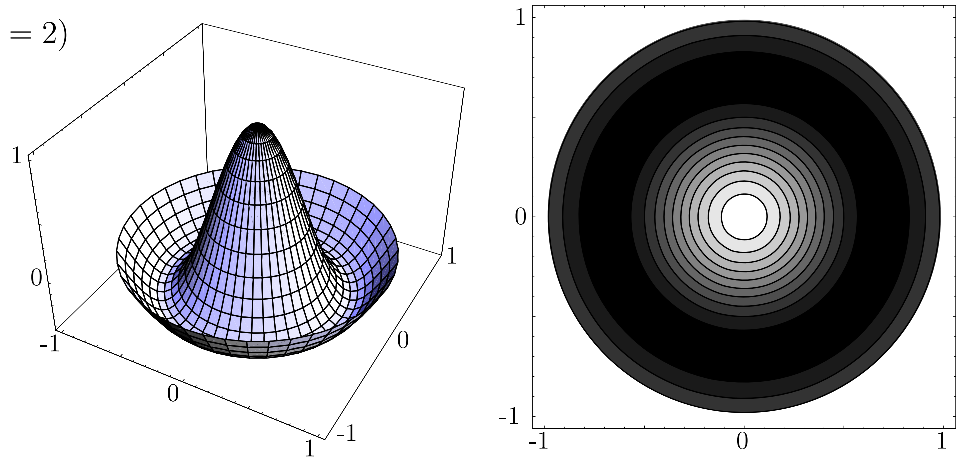




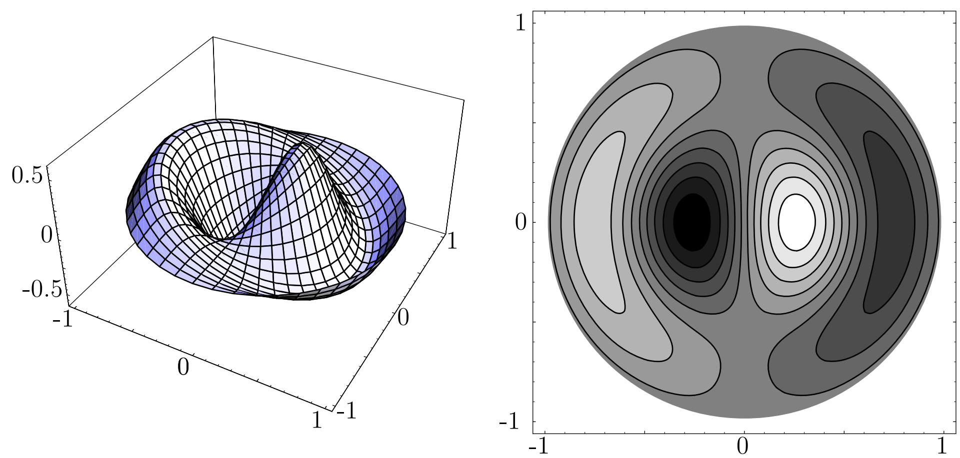
Vibrational modes of a timpani ($m = 2$)

$$J_n\left(\frac{z_{mn} r}{a}\right)\cos(n\theta)\cos\left(\frac{c z_{mn} t}{a}\right)$$

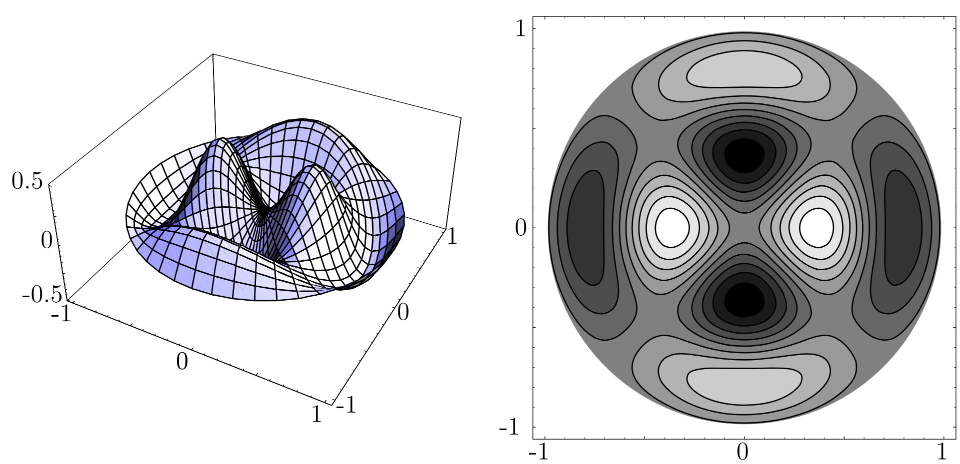
$(m, n) = (2, 0)$



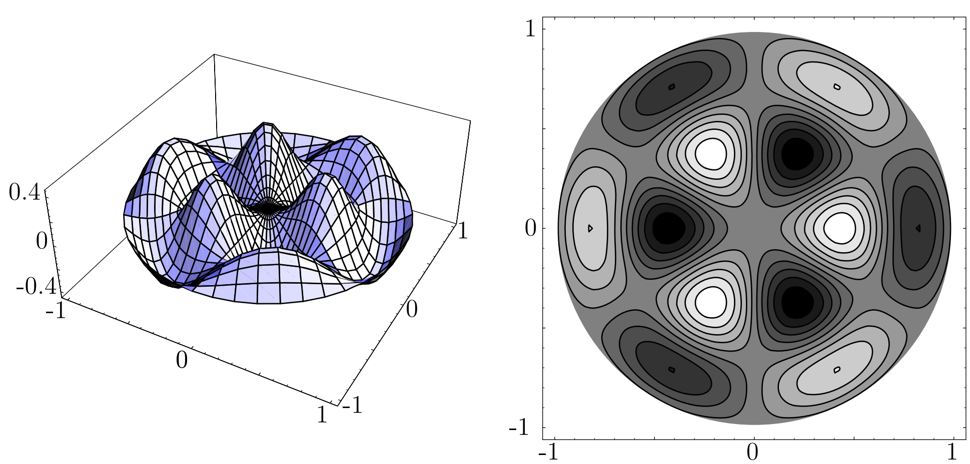
$(m, n) = (2, 1)$



$(m, n) = (2, 2)$



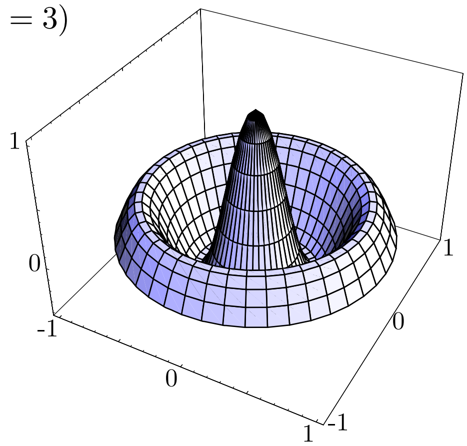
$(m, n) = (2, 3)$

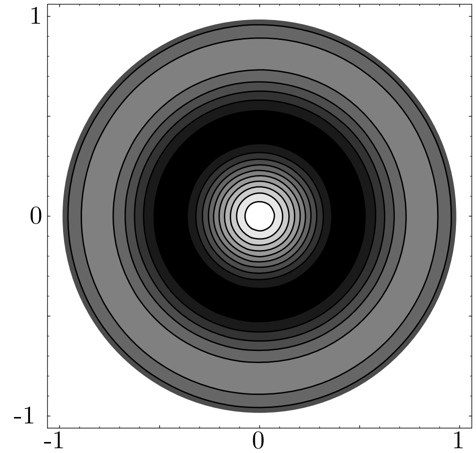


Vibrational modes of a timpani ($m = 3$)

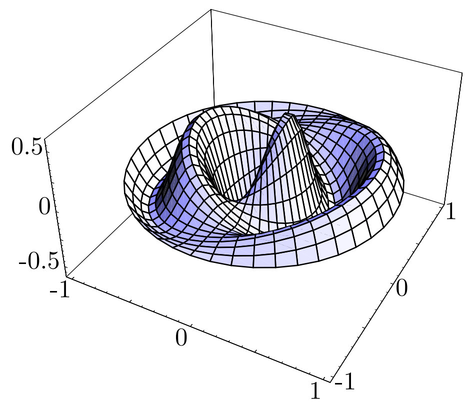
$$J_n\left(\frac{z_{mn} r}{a}\right) \cos(n\theta) \cos\left(\frac{c z_{mn} t}{a}\right)$$

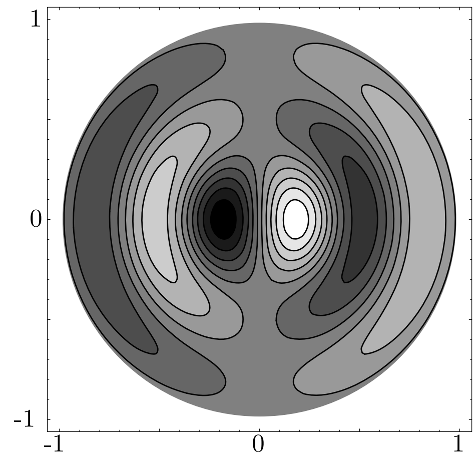
$(m, n) = (3, 0)$



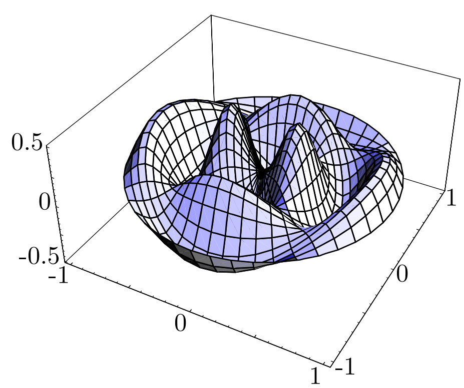


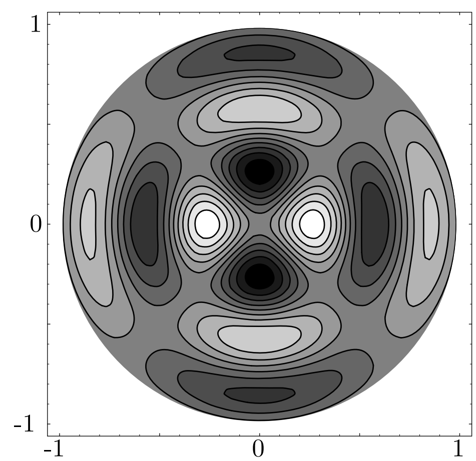
$(m, n) = (3, 1)$



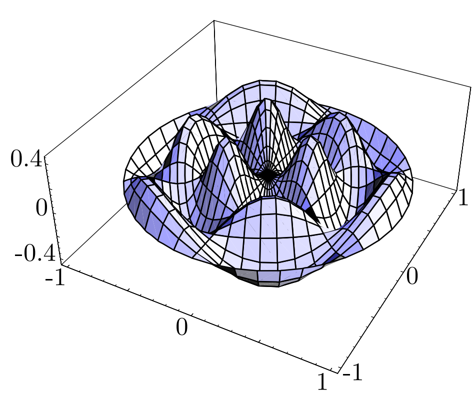


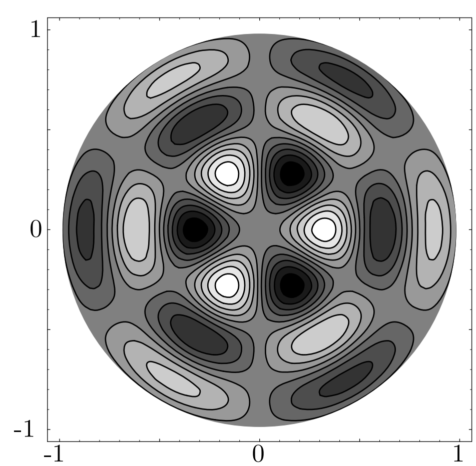
$(m, n) = (3, 2)$





$(m, n) = (3, 3)$





Equal Temperament Tuning

Note Name	Frequency (Hz)	Ratio to C4
A_3	220.00	
$A\#_3$	233.08	≈ 8/9
B_3	246.94	
C_4	261.63	1
$C\#_4$	277.18	
D_4	293.66	≈ 9/8
$D\#_4$	311.13	
E_4	329.63	≈ 81/64
F_4	349.23	≈ 4/3
$F\#_4$	369.99	
G_4	392.00	≈ 3/2
$G\#_4$	415.30	
A_4	440.00	≈ 27/16
$A\#_4$	466.16	
B_4	493.88	≈ 243/128
C_5	523.25	2

ACM 95/100c notes: Vibrational modes of a timpani
Darryl Yong SDG 4/4/2003

Mathematica's built-in function for finding the roots of the BesselJ function is BesselJZeros, but it uses its arguments in a slightly different way so I defined my own BJZeros function. BJZeros[m,n] gives the mth positive zero of the Bessel function of the first kind, order n.

```
<< NumericalMath`BesselZeros`

temp = Table[BesselJZeros[i, 20], {i, 0, 10}];
BJZeros[m_, n_] := temp[[n + 1, m]]
```

■ This is a table of frequency ratios of higher vibrational modes to the lowest (1,0) vibrational mode.

```
Table[BJZeros[m, n] / BJZeros[1, 0], {n, 0, 3}, {m, 1, 4}]
```

1.	2.29542	3.59848	4.90328
1.59334	2.9173	4.23044	5.5404
2.13555	3.50015	4.83189	6.15261
2.65307	4.05893	5.41212	6.74621

```
Table[220 * BJZeros[m, n] / BJZeros[1, 0], {n, 0, 3}, {m, 1, 4}]
```

220.	504.992	791.667	1078.72
350.535	641.805	930.697	1218.89
469.821	770.032	1063.01	1353.57
583.675	892.965	1190.67	1484.17

■ This is a table of frequency ratios of all vibrational modes to the (1,1) vibrational mode.

```
Table[BJZeros[m, n] / BJZeros[1, 1], {n, 0, 3}, {m, 1, 4}]
```

0.627612	1.44063	2.25845	3.07736
1.	1.83093	2.65508	3.47722
1.3403	2.19674	3.03255	3.86145
1.6651	2.54744	3.39671	4.23401

```
Table[220 * BJZeros[m, n] / BJZeros[1, 1], {n, 0, 3}, {m, 1, 4}]
```

138.075	316.939	496.86	677.019
220.	402.805	584.117	764.989
294.865	483.282	667.161	849.52
366.321	560.436	747.277	931.481

■ Discussion

If one hits the timpani exactly in the center, the (1,0), (2,0), (3,0), etc. modes are preferentially excited. No modes (m,n) with n>0 are excited because these modes all have node lines that cross at the center of the timpani. Because the energy is directed into a small set of modes, the sound is somewhat hollow. Furthermore, the ratio of the frequency of the (2,0) mode to the (1,0) mode is about 2.29542, so if the (1,0) mode is tuned to A3 at 220 Hz, then the (2,0) mode sounds at about 504.992 Hz, which is between B4 and C5 (a very discordant musical interval).

Any good percussionist (aka Joe Jewell) will tell you that the best place to hit the timpani is along a circle with about 2/3 or 3/4 the radius of the entire membrane (about 3 to 4 inches in from the edge of the timpani). In his classic book "The Theory of Sound" (1877), Lord Rayleigh observed that when the timpani is struck at this spot, the (1,0) mode is not excited, which can be confirmed by the following experiment: Place a small pile of small particles (like iron fillings or pepper) at the center of the drumhead, and observe that when the timpani is struck at the right spot (the one Joe recommends), the pile of particles does not move very much. Lord Rayleigh says that when the timpani is struck in this way, the principal vibration is the (1,1) mode.

The mathematical analysis also seems to suggest that this excitation of modes produces a more pleasing sound, because the ratio of the frequency of the (1,2) mode to the (1,1) mode is about 1.3403, so if the (1,1) mode is tuned to A3 at 220 Hz, then the (1,2) mode sounds at about 294.865 Hz, which is close to D4, forming a perfect 4th (a consonant musical interval). In practice, there are many other factors* which contribute to the sounds that actually reach your ear, and the interval formed between the (1,1) and (1,2) modes is closer to a perfect fifth (ratio of 1.5).

* Here are a few things that we have neglected in this analysis. First, and most importantly, the motion of the timpani is damped, which changes the vibration modes and their frequencies. Second, there are nonlinear effects (such as surface tension), which exert their own preference for certain modes by transferring energy from one vibration mode to another. Third, the tension of the drumhead is not uniform across the entire drumhead (c is not really constant). Furthermore, there is a nonlinear coupling between the vibrations of the membrane, the vibrations of the copper bowl, and the vibrations of the air particles that eventually reach your ear.

■ References:

Backus, John, The Acoustical Foundations of Music, 2nd Ed, W W Norton, New York, 1977
Benade, Arthur H., Fundamentals of Musical Acoustics, Oxford University Press, 1976