Modeling Faraday Excitation of a Viscous Fluid

by

Bradley Forrest

Andrew Bernoff, Advisor

Advisor: ______________________________

Second Reader: ______________________________

(T. D. Donnelly)

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Department of Mathematics

HARVEY MUDD COLLEGE

Abstract

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Faraday Excitation is the occurrence of growing surface waves when a fluid is subjected to periodic forcing. Given a fluid, Faraday Excitation will occur for some, but not all, values of forcing frequency and amplitude. In this thesis, a viscous fluid is modeled through linear stability analysis of the Navier-Stokes equation and predictions for the conditions when Faraday Excitation will occur are given. Predictions for the scaling between excitation wavelength and forcing frequency are also given. This information will be used in future work to predict droplet size for an experiment using ultrasound to produce aerosols being conducted by T. D. Donnelly's research group. For fluids with low non dimensional wave number the scaling between excitation wave number, k, and the forcing frequency ω was determined to be $k = .62616(\sigma/\rho)^{\frac{-1}{3}}\omega^{\frac{2}{3}}$ and $k = .47206\nu^{\frac{-1}{2}}\omega^{\frac{1}{2}}$ for high non dimensional wave number.

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Acknowledgments

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Chapter 1

Introduction

The study of fluid motion is a rich subject requiring both mathematical acumen and physical intuition. Many basic intuitive facts about fluid flow are challenging to confirm mathematically. Similarly, mathematical models of fluids often shed light on unforeseen and unintuitive behaviors. This thesis examines the behavior of a fluid subjected to an oscillating vertical acceleration by utilizing mathematical modeling methods. In some cases, an oscillating fluid exhibits surface waves. This behavior is known as Faraday Excitation. The main goal of this thesis is to predict the relationship between the wavelength of surface waves and the acceleration's oscillation frequency for any excited fluid. This relationship has an important role in ultrasonic atomization, which is discussed in Section 1.1. A review of current literature on Faraday Excitation is given in Section 1.2. Section 1.3 explains the basic structure and assumptions used by the mathematical models in describing the physical system. The last section of this chapter describes the content of each of the following chapters.

1.1 Connection to Ultrasonic Atomization and Droplet Experimentation

In some cases of Faraday Excitation, the excited surface waves grow in amplitude over time. This amplitude growth causes the fluid surface to stretch, and eventually to become unstable. At this point, a droplet is ejected from the apex of the surface wave. This droplet ejection is referred to as ultrasonic atomization in the case of sufficiently small droplets. Over many experiments, it has been established that the radius of the ejected particle varies only by a constant from the wavelength of the fluid surface waves. [12] [15] [5] [6] Thus, the relationship examined in this work can be used to predict the size of ejected droplets.

In ultrasonic atomization literature, the value of the constant relating the wavelength of an excited surface wave to the radius of a droplet ejected by that wave has not been determined with significant precision for deep fluid waves. With improved precision due to MIE scattering particle sizing techniques, T. D. Donnelly's experimental physics group has been measuring the size of droplets made through Faraday Excitation. Combining these measurements with the predictions made in this work will yield a more precise value for the particle sizing constant, and these results will be put together in future work.

1.2 Literature Review

Faraday Excitation is not unfamiliar ground for fluid dynamics researchers. In 1954, Benjamin and Ursell showed that the surface of an ideal fluid under excitation could be successfully modeled with the Mathieu Equation.[1] In this analysis, Benjamin and Ursell assumed that the fluid was weakly viscous, that the fluid's viscosity could be treated as as a small perturbation. A full treatment of Faraday Excitation for viscous fluids requires the application of numerical methods. Only in the last 10-15 years has the computing speed necessary to carry such computations become widely available. In 1994, Kumar and Tuckerman present a linear stability analysis of viscous excitation of two fluids. [9] Parameter regions for surface instability, growing surface waves, are calculated for an ideal fluid and an example viscous fluid. Beyer showed that the fully viscous problem could be reduced to a modified Mathieu equation. [3] Besson and Weizhong each extended Kumar and Tuckerman's work to case of two frequency forcing, and successfully tested these theoretical predictions against experimental results. [2] [16] Many researchers have found new ways to formulate models of Faraday Excitation. In 1997, Cerda showed a relationship between Faraday Excitation and Rayleigh-Taylor instability. [4] Miles formulated the Faraday problem in terms of the impedance of the liquid. [13]

In this work, the original linear stability analysis of this problem is used to predict the relationship between the wavelength of surface waves and the forcing frequency. In the body of work on Faraday Excitation, this relationship has not been explicitly calculated, but the power laws that relate the parameters can be solved for through dimensional reasoning. It is the constants in these equations that are the goal of this work, as these constants will be necessary in determining the constant that relates droplet size to surface wavelength as described in Section 1.1.

1.3 Description of Physical Situation and Assumptions About Fluid

Consider a viscous fluid in a deep container, subjected to an oscillating vertical forcing acceleration $a\cos(\omega t)$.

The models in this work assume that the container is infinitely deep, which is an accurate approximation when the container depth is much greater than the wavelength of surface waves. In order to describe this fluid, assume that the fluid has the following constant fluid parameters:

Physical Fluid Parameters

$$\begin{aligned} \nu &\equiv \text{kinematic viscosity} \\ \sigma &\equiv \text{surface tension} \\ \rho &\equiv \text{density} \end{aligned}$$

In most cases, these parameters are constant for a fluid, assuming that the fluid is not subjected to a temperature gradient. Additionally, to describe the forcing and wavelength of the surface waves, define constant control parameters as follows:

Control Parameters

$$\begin{aligned} \omega &\equiv \text{forcing frequency} \\ a &\equiv \text{input acceleration} \\ k &\equiv \text{wave number of surface waves} = 2\pi/\lambda, \end{aligned}$$

where λ is the wavelength of the surface waves. In this thesis, the fluid is treated as two-dimensional. Additionally, effects of the edges of the fluid container have been neglected, and hence it is assumed that the horizontal dimension is infinite. Also, the the effect of gravity has been neglected.

1.4 The Chapters Ahead

There is substantial background knowledge necessary before the linear stability analysis for an excited viscous fluid can be understood. The first few chapters build the necessary background information. Chapter 2 introduces Floquet Theory, which is an indispensable tool utilized in investigating the ordinary and partial differential equations the arise naturally in the stability analysis. In Chapter 3, Faraday Excitation of weakly viscous fluids is examined through stability analysis of the Mathieu Equation. Parameter regions yielding growing solutions are given. Chapter 4 details how Fourier methods can be used to enhance the stability analysis shown in Chapter 3. In Chapter 5, the equations needed for the linear stability analysis of an excited highly viscous fluid are derived from first principles. These equations, which include the Navier-Stokes equation, kinematic condition, and surface force balance equation, are known collectively as the hydrodynamic system. Chapter 6 explores solutions to the hydrodynamic system without including forcing. In Chapter 7, solutions to the hydrodynamic system are investigated with forcing included. Additionally, predictions of the wave number's scaling with the forcing frequency for both high and low viscosity are given. Concluding commentary is given in Chapter 8.

Chapter 2

Floquet Theory

Before attacking the physical problem, it is necessary to develop the tools that will aide in solving the ordinary and partial differential equations that occur in the problem. One such tool is Floquet theory, which applies linear algebra methods to a particular matrix system of differential equations. In this chapter, several important theorems in Floquet theory are proved. The proof presented in this chapter are adapted from those given by Jordan in *Nonlinear Ordinary Differential Equations.* [8]

Consider a system of ODEs: $\frac{dZ}{dt} = P(t)Z$ where $Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, and $P(t)$ is a $N \times N$ matrix with minimum period τ.

Definition 1: Let $\phi_1(t), \phi_2(t), \ldots \phi_n(t)$, be linearly independent solutions to the system described above. Then $\Phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))$, the $N \times N$ matrix with these solutions as its columns, is known as a fundamental matrix.

Theorem 2.0.1 (Floquet's Theorem) *The regular system $\frac{dZ}{dt} = P(t)Z$ where $P(t)$ is a $N \times N$ matrix with minimum period τ, has a non trivial solution $Z = \chi(t)$ such that $\chi(t + \tau) = \mu\chi(t)$, where μ is a constant.*

Proof: Let $\Phi(t)$ be a fundamental matrix of the system. This is a matrix with a set of linearly independent solutions to the system as its columns. Note that

$\frac{d\Phi}{dt}(t)= P(t)\Phi(t)$, and $\frac{d\Phi}{dt}(t+\tau) = P(t+\tau)\Phi(t+\tau)$. Since P is periodic $P(t) = P(t+\tau)$, we have $\frac{d\Phi}{dt}(t+\tau) = P(t)\Phi(t+\tau)$. $\Phi(t+\tau)$ is also a fundamental matrix of the system and hence the columns of $\Phi(t+\tau)$ can be written as linear combinations of the columns of $\Phi(t)$. Another way to think about this is that the columns of $\Phi(t)$ are a basis for the solution space of the system of ODEs. Hence, the solutions that make up $\Phi(t+\tau)$ can be written in this basis. Let $\Phi(t)= \phi_{ij}(t)$, and hence $\phi_{ij}(t+\tau) = \sum_{k=1}^{n} \phi_{ik}(t)e_{kj}$, where e_{kj} represents a matrix of constants, let this E be this matrix. Thus, $\Phi(t+\tau) = \Phi(t)E$. Note that since $\Phi(t)$, and $\Phi(t+\tau)$ are fundamental matrices, they are both non singular, thus $\det(\Phi(\mathrm{t})) \neq 0$ and $\det(\Phi(\mathrm{t}+\tau)) \neq 0$. This implies that E is non singular as $\det(\Phi(\mathrm{t})) = \det(\Phi(\mathrm{t}+\tau)\det(\mathrm{E})$, meaning that $\det(\mathrm{E}) \neq 0$.

Consider μ an eigenvalue of E, and S, $\mu' s$ eigenvector. Therefore, $(E - I\mu)S = 0$, and $\mu S = ES$. Let $\chi(t) = \Phi(t)S$. $\chi(t)$ is a linear combination of basis vectors of the solution space of the system. Thus, $\chi(t)$ is also a solution vector. Further

$$\begin{aligned}
\chi(t+\tau) &= \Phi(t+\tau)S \\
&= \Phi(t)ES \\
&= \Phi(t)\mu S \\
&= \mu\chi(t)
\end{aligned}$$

□

Definition 2: A solution satisfying the equation $\chi(t+\tau) = \mu\chi(t)$ is known as a normal solution.

Definition 3: The eigenvalues, μ_i of the matrix E as defined above are known as the characteristic numbers of the system.

Definition 4: Let $e^{\rho_i \tau} = \mu_i$. The ρ_i's are known as the characteristic exponents of the system.

Theorem 2.0.2 *Let E be an $N \times N$ matrix such that $\Phi(t+\tau) = \Phi(t)E$ where $\Phi(t)$ is the fundamental matrix of a regular system of ODEs that meets the conditions of Theorem 1. If E has n distinct eigenvalues, $\mu_1, ..., \mu_n$, then the system has n linearly independent normal solutions of the form $Z_i = p_i(t)e^{\rho_i t}$ where $\mu_i = e^{\rho_i \tau}$ and the $p_i(t)$ are functions with period τ.*

Proof: For each μ_i there exists a solution $x_i(t)$ satisfying $x_i(t+\tau) = \mu_i x_i(t) = e^{\rho_i \tau} x_i(t)$. Hence, $e^{-\rho_i t} x_i(t+\tau) = e^{\rho_i(\tau - t)} x_i(t)$, and further $e^{-\rho_i(t+\tau)} x_i(t+\tau) = e^{\rho_i t} x_i(t)$. Letting, $p_i(t) = e^{-\rho_i t} x_i(t)$, we see that $p_i(t)$ is periodic and $x_i(t) = p_i(t)e^{\rho_i t}$. □

Theorem 2.0.3 (Abel's Theorem) *Let $\Phi(t)$ be a matrix of solutions of $\frac{\mathbf{dZ}}{\mathbf{dt}} = P(t)$ Z. Then for any time t_o, $W(t) = W(t_o)e^{\int_{t_o}^{t} tr(P(s))ds}$. In this equation, $W(t) =$ $\det(\Phi(\mathrm{t}))$ is the Wronksian of the set of solutions that comprise $\Phi(t)$, and $tr(P(t))$ is the trace of $P(t)$.*

Proof: If the solutions that make up $\Phi(t)$ are linearly independent, then $\det(\Phi(\mathrm{t})) = 0$, and so $W(t) = W(t_o) = 0$.

If the solutions are linearly independent then $\Phi(t)$ is a fundamental matrix of the system, let $\Phi(t) = (\phi_{ij}(t))$. Let $\Delta_k = \Phi(t)$ except for the kth row which is replaced by $\frac{d\phi_{kj}}{dt}(t)$. Lets examine $\frac{dW}{dt}(t) = \frac{d(\det(\Phi))}{dt}(t)$. This is equivalent to the sum of the determinants of the n δ_k matrices, $\sum_{k=1}^{n} \det(\delta_\mathrm{k})$. For notational simplicity, we will examine the case when $\Phi(t)$ is a 2×2 matrix, since the proof is easily generalizable.

$$\frac{dW}{dt}(t) = \sum_{k=1}^{2} \det(\delta_\mathrm{k}) = \det(\delta_1) + \det(\delta_2) = \begin{vmatrix} \frac{d\phi_{11}}{dt} & \frac{d\phi_{12}}{dt} \\ \phi_{21} & \phi_{22} \end{vmatrix} + \begin{vmatrix} \phi_{11} & \phi_{12} \\ \frac{d\phi_{21}}{dt} & \frac{d\phi_{22}}{dt} \end{vmatrix}$$

Note that $\frac{d\Phi(t)}{dt} = P(t)\Phi(t)$, so $\frac{d\phi_{ij}}{dt}(t) = \sum_{k=1}^{2} p_{ik}\phi_{kj}(t)$. Thus

$$\begin{vmatrix} \frac{d\phi_{11}}{dt} & \frac{d\phi_{12}}{dt} \\ \phi_{21} & \phi_{22} \end{vmatrix} + \begin{vmatrix} \phi_{11} & \phi_{12} \\ \frac{d\phi_{21}}{dt} & \frac{d\phi_{22}}{dt} \end{vmatrix} =$$

$$\begin{vmatrix} \sum_{k=1}^{2} p_{1k}\phi_{k1} & \sum_{k=1}^{2} p_{1k}\phi_{k2} \\ \phi_{21} & \phi_{22} \end{vmatrix} + \begin{vmatrix} \phi_{11} & \phi_{12} \\ \sum_{k=1}^{2} p_{2k}\phi_{k1} & \sum_{k=1}^{2} p_{2k}\phi_{k2} \end{vmatrix} =$$

$$p_{11}\begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{vmatrix} + p_{12}\begin{vmatrix} \phi_{21} & \phi_{22} \\ \phi_{21} & \phi_{22} \end{vmatrix} + p_{21}\begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{11} & \phi_{12} \end{vmatrix} + p_{22}\begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{vmatrix} =$$

$$p_{11}W(t) + 0 + 0 + p_{22}W(t) = (p_{11} + p_{22})W(t) = tr(P(t))W(t)$$

Hence, $\frac{dW}{dt}(t) = tr(P(t))W(t)$, which is the differential equation for $W(t)$ that yields the solution $W(t) = W(t_o)e^{\int_{t_o}^{t} tr(P(s))ds}$. □

Theorem 2.0.4 *For the system* $\frac{dZ}{dt} = P(t)Z$*, where* $P(t)$ *has minimum period* τ*, let* $\mu_1\mu_2\ldots\mu_n$ *be the characteristic numbers of the system. Then* $\mu_1\mu_2\ldots\mu_n = e^{\int_0^{\tau} tr(P(s))ds}$.

Proof: Let $\Psi(t)$ be the fundamental matrix with initial condition $\Psi(0) = I$. Then for matrix E as defined in the proof of theorem 2.1.1, $\Psi(\tau) = \Psi(0)E = E$. Recalling that the characteristic numbers are the eigenvalues of E, $\det(\mathrm{E} - \mu\mathrm{I}) = (\mu_1 - \mu)(\mu_2 - \mu)\ldots(\mu_\mathrm{n} - \mu)$, when $\mu = 0$, $\det(\mathrm{E}) = \mu_1\mu_2\ldots\mu_\mathrm{n}$. Hence, $\mu_1\mu_2\ldots\mu_n = \det(\mathrm{E}) =$

$\det(\Psi(\tau)) = \mathrm{W}(\tau)$. By applying Theorem 2.1.3, with $t = \tau$, and $t_o = 0$ we find that $\mu_1\mu_2 \ldots \mu_n = W(\tau) = W(0)e^{\int_0^\tau tr(P(s))ds} = e^{\int_0^\tau tr(P(s))ds}$. $\square$

Chapter 3

Mathieu Model

This chapter presents the Mathieu model for the physical system. It should be noted that this model is only valid for a weakly viscous fluid. Hence, results from the Mathieu model yield an asymptotic expectation to check more general results against, but are not in and of themselves productive.

3.1 *Equation Introduction*

Assuming that the forcing is not over-damped by the fluid's viscosity, wave excitation will occur. In particular, it will occur in the form $\eta(x,t) = \zeta(t)e^{ikx}$, where η is the height above the fluid level at rest, and x, the horizontal dimension, can be removed in this way because this dimension is infinite. Thus any physical phenomenon must be symmetric with respect to this dimension.

For weakly viscous fluids, the following equation governs this wave motion:

$$\frac{d^2\zeta}{dt^2} + 4\nu k^2 \frac{d\zeta}{dt} + (k^3\sigma/\rho - ak\cos(\omega t))\zeta = 0. \tag{3.1}$$

This is the damped Mathieu Equation, which describes the wave motion under the deep fluid and weak viscous effects limit.[10]

It is helpful to consider the analog of a damped harmonic oscillator in order to understand the different terms in Equation (3.1). The equation governing the motion of a damped harmonic oscillator with damping γ, and forcing amplitude $\xi\cos(\omega t)$ is:

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + (\Omega^2 - \xi\omega^2\cos(\omega t))\theta = 0,$$

where Ω is the oscillator's natural oscillation frequency, and θ is the angle of the oscillator from rest. From this analog, we can see that the fluid's viscosity acts to damp wave oscillations while the surface tension and density set the natural oscillation frequency of the fluid.

3.2 Non-Dimensionalization and DE Simplification

To make this equation easier to handle, it is helpful to non-dimensionalize the coefficients. Non-dimensionalizing these coefficients is also a good idea because it allows simple comparisons to made between different physical experiments.

Non-dimensionalizing is accomplished by scaling time, letting $T = \omega t$, which gives $\omega \frac{d}{dT} = \frac{d}{dt}$ and $\omega^2 \frac{d}{dT^2} = \frac{d}{dt^2}$. Applying these substitutions and dividing through by ω^2 yields:

$$\frac{d^2\zeta}{dT^2} + (4\nu k^2/\omega)\frac{d\zeta}{dT} + (k^3\sigma/\rho\omega^2 - (ak/\omega^2)\cos(T))\zeta = 0. \tag{3.2}$$

This equation is not solvable by any standard means, so we will pursue a numerical approximation. Before applying any such numerical solver, the ODE must be simplified significantly. The first step in simplifying this differential equation is applying the substitution:

$$ze^{-2\nu k^2 T/\omega} = \zeta. \tag{3.3}$$

This change of variables gives the following:

$$\frac{d^2 z}{dT^2} + (k^3\sigma/\rho\omega^2 - 4\nu^2 k^4/\omega^2 - (ak/\omega^2)\cos(T))z = 0. \tag{3.4}$$

The damping and oscillation terms from Equation (3.2) have been coupled to create Equation (3.4). The oscillation term in Equation (3.4) accounts for the removed damping. Letting

$$A = k^3\sigma/\rho\omega^2 - 4\nu^2k^4/\omega^2,$$

and

$$B = -ak/\omega^2$$

yields the following simplification:

$$\frac{d^2z}{dT^2} + (A + B\cos(T))z = 0. \tag{3.5}$$

This equation is now only dependent upon the parameters A and B. In this form, the solutions to the equation can be investigated using a numerical solver for points on the AB plane.

3.3 *Application of Floquet Theory*

Equation (3.5) can be written as a system of ODEs, as follows:

$$\begin{bmatrix} \frac{dz_1}{dT} \\ \frac{dz_2}{dT} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -A - B\cos(T) & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

This is a specific case of a more general form of the ODE system

$$\frac{d\bar{Z}}{dT} = P(T)\bar{Z}(T), \tag{3.6}$$

where

$$\bar{Z}(T) = \begin{bmatrix} z_1(T) \\ z_2(T) \end{bmatrix}$$

and $P(T)$ is a matrix with minimum period τ, which is 2π in the specific case of the Mathieu Equation. For ODEs of this form, results from Floquet Theory can be applied to help characterize solutions.

Theorem 2.1.2 shows that the solutions to this system are of the form

$$\bar{Z}_i(T) = p_i(T)e^{\alpha_i T}, \tag{3.7}$$

where $p_i(T)$ is some τ periodic column vector, and the subscript i denotes the existence of multiple linearly independent solutions. For each solution, α_i is known as the characteristic exponent. To each characteristic exponent, there corresponds a characteristic number, μ_i, defined as

$$\mu_i = e^{\alpha_i \tau}. \tag{3.8}$$

Theorem 2.1.4 relates the product of the characteristic numbers of the system to the trace of the coefficient matrix $P(T)$ in the following manner:

$$\mu_1 \mu_2 \ldots \mu_n = e^{\int_0^\tau tr(P(S))dS}. \tag{3.9}$$

Applying these results to the specific case of the Mathieu Equation, first we note that the Mathieu Equation has only two linearly independent solutions, and thus only two characteristic numbers. Applying Equation (3.9) yields,

$$\mu_1 \mu_2 = 1, \tag{3.10}$$

and hence,

$$\mu_2 = 1/\mu_1.$$

3.4 Time Reversal Symmetry

A property of the Mathieu Equation that will prove helpful is that it is time reversible. To see this note that all of the coefficients in $P(t)$ are even functions. Given $\bar{T} = -T$, a solution to the system $\begin{bmatrix} z_1(T) \\ z_2(T) \end{bmatrix}$ when transformed to $\bar{T}$ becomes $\begin{bmatrix} z_1(\bar{T}) \\ -z_2(\bar{T}) \end{bmatrix}$. Hence,

$$\begin{bmatrix} z_1(T) \\ z_2(T) \end{bmatrix} = \begin{bmatrix} z_1(\bar{T}) \\ -z_2(\bar{T}) \end{bmatrix} \tag{3.11}$$

To verify that this is the correct time reversal symmetry, lets examine this transformed system.

$$\begin{bmatrix} -\frac{dz_1}{d\bar{T}} \\ \frac{dz_2}{d\bar{T}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -A - B\cos(T) & 0 \end{bmatrix} \begin{bmatrix} z_1(\bar{T}) \\ -z_2(\bar{T}) \end{bmatrix}$$

When written as a single ODE, this transformation gives:

$$\frac{d^2 z_1}{d\bar{T}^2} + (A + B\cos(\bar{T}))z_1 = 0,$$

which is the same ODE as Equation (3.5), verifying the symmetry.

3.5 Introduction of Matrix E

Let $\Phi(T)$ be a fundamental matrix of the Mathieu Equation, and $\phi_1(t)$, and $\phi_2(t)$ as its columns. Note that by Equation (3.6):

$$\frac{d\Phi}{dt}(t) = P(t)\Phi(t),$$

and

$$\frac{d\Phi}{dt}(t+\tau) = P(t+\tau)\Phi(t+\tau)$$

since fundamental matrices have solutions as their columns. Since P is periodic

$$P(t) = P(t+\tau),$$

by substitution we have

$$\frac{d\Phi}{dt}(t+\tau) = P(t)\Phi(t+\tau).$$

$\Phi(t+\tau)$ is also a fundamental matrix of the system as its columns are solutions of Equation (3.6). Since the columns of $\Phi(t)$ are a basis for the solution space of the Mathieu Equation, the solution columns of $\Phi(t+\tau)$ can be written as linear combinations of the columns of $\Phi(t)$. Let

$$\Phi(t) = \phi_{ij}(t),$$

hence

$$\phi_{ij}(t+\tau) = \sum_{k=1}^{n} e_{ik}\phi_{kj}(t),$$

where e_{ik} represents a matrix of constants, let E be this matrix. Therefore,

$$E\Phi(T) = \Phi(T+\tau), \tag{3.12}$$

for all T. Note that this is also true for any solution vector $z(T)$,

$$Ez(T) = z(T+\tau). \tag{3.13}$$

3.6 Finding the Eigenvectors of E

The characteristic numbers of the Mathieu Equation are the eigenvalues of E and μ_1 and $1/\mu_1$ are the eigenvalues for E. Let $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ be the eigenvector for μ_1 and let

$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ be the eigenvector for $1/\mu_1$. Thus,

$$E \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \mu_1 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

and

$$E \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = (1/\mu_1) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

These eigenvectors must be linearly independent, and it follows that any solution vector's initial condition can be written as a linear combination of these eigenvectors. Let $\bar{z}(T) = \begin{bmatrix} z_1(T) \\ z_2(T) \end{bmatrix}$ be a solution with initial condition

$$\bar{z}(0) = a_o \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + b_o \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Then, by taking $T = 0$ in Equation (3.13)

$$\begin{aligned} \bar{z}(\tau) &= \begin{bmatrix} z_1(\tau) \\ z_2(\tau) \end{bmatrix} \\ &= E\bar{z}(0) \\ &= a_o E \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + b_o E \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\ &= a_o \mu_1 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + b_o(1/\mu_1) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\ &= \begin{bmatrix} a_o\mu_1 e_1 + b_o(1/\mu_1) f_1 \\ a_o\mu_1 e_2 + b_o(1/\mu_1) f_2 \end{bmatrix} \end{aligned}$$

Also, note that $E\bar{z}(-\tau) = \bar{z}(0)$, and hence

$$\begin{aligned}
\bar{z}(-\tau) &= \begin{bmatrix} z_1(-\tau) \\ z_2(-\tau) \end{bmatrix} \\
&= E^{-1}\bar{z}(0) \\
&= a_o E^{-1} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + b_o E^{-1} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\
&= a_o(1/\mu_1) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + b_o \mu_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\
&= \begin{bmatrix} a_o(1/\mu_1)e_1 + b_o\mu_1 f_1 \\ a_o(1/\mu_1)e_2 + b_o\mu_1 f_2 \end{bmatrix}.
\end{aligned}$$

By Equation (3.11), $z_1(\tau) = z_1(-\tau)$ and $z_2(\tau) = -z_2(-\tau)$. Substituting for the z_1, and z_2's, yields

$$a_o\mu_1 e_1 + b_o(1/\mu_1)f_1 = a_o(1/\mu_1)e_1 + b_o\mu_1 f_1 \tag{3.14}$$

and

$$a_o\mu_1 e_2 + b_o(1/\mu_1)f_2 = -a_o(1/\mu_1)e_2 - b_o\mu_1 f_2. \tag{3.15}$$

These equations yield that

$$a_o/b_o \begin{bmatrix} e_1 \\ -e_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Therefore,

$$(a_o/b_o)E \begin{bmatrix} e_1 \\ -e_2 \end{bmatrix} = (a_o/b_o)(1/\mu_1) \begin{bmatrix} e_1 \\ -e_2 \end{bmatrix},$$

and

$$E \begin{bmatrix} e_1 \\ -e_2 \end{bmatrix} = (1/\mu_1) \begin{bmatrix} e_1 \\ -e_2 \end{bmatrix}.$$

$\begin{bmatrix} e_1 \\ -e_2 \end{bmatrix}$ is an eigenvector of $1/\mu_1$. [8]

3.7 Method of Stability Analysis

Knowing the eigenvectors in terms of one another will help to find an equation for μ_1, which as seen in equations (7) and (8) will tell us the growth rate of the solution. To that end, consider the initial condition for the solution $z(T)$, $z(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Note that this vector can be re-written such that,

$$z(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1/2e_1)(\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ -e_2 \end{bmatrix}).$$

Applying Equation (3.12) yields,

$$\begin{aligned} z(\tau) &= \begin{bmatrix} z_1(\tau) \\ z_2(\tau) \end{bmatrix} = E \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= (1/2e_1)E(\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ -e_2 \end{bmatrix}) \\ &= (1/2e_1)(\mu_1 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + (1/\mu_1) \begin{bmatrix} e_1 \\ -e_2 \end{bmatrix}) \\ &= \begin{bmatrix} (\mu_1 + 1/\mu_1)/2 \\ 0 \end{bmatrix}. \end{aligned}$$

Hence,

$$z_1(\tau) = (\mu_1 + 1/\mu_1)/2 \tag{3.16}$$

and through the use of the quadratic equation,

$$\mu_1 = z_1(\tau) \pm ((z_1(\tau))^2 - 1). \tag{3.17}$$

A numerical approximation can be applied to find $z_1(\tau)$ given values for A and B, and with the initial condition $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The numerically approximated value for $z_1(\tau)$ can be substituted into Equation (3.17) to find a value for μ_1.

Now the value of $|z_1(\tau)|$ determines the stability of the solution. The discussion of this was adapted from Jordan's text. [8] Note that for $|z_1(\tau)| < 1$, μ_1, μ_2 are complex, while $|z_1(\tau)| \geq 1$ μ_1 and μ_2 are real. The two characteristic numbers correspond to the two different solutions of the differential equation. The form of these solutions is given in Equation (3.7). The solutions have two terms, a periodic term and a growth term. The characteristic number relates to the growth terms as shown in Equation (3.8).

If the characteristic numbers are complex, they must be complex conjugates and must have magnitude 1 since $\mu_1\mu_2 = 1$. In this regime, the solutions are

$$\begin{aligned} z_1(T) &= c_1 e^{\beta i T} p_1(T) + c_2 e^{-\beta i T} p_2(T) \\ &= c_1 p_1(T)(\cos(\beta T) + i\sin(\beta T)) + c_2 p_2(T)(\cos(-\beta T) + i\sin(-\beta T)). \end{aligned}$$

These solutions are oscillatory, but are not generally periodic as they contain terms with periods 2π and $2\pi/\beta$. These solutions are bounded.

If $z_1(\tau) = 1$, then $\mu_1 = \mu_2 = 1$, and the characteristic exponents are 0. Hence, there is one solution of the form $z_1 = p_1(T)$, and thus there exists one 2π periodic solution.

If $z_1(\tau) = -1$, then $\mu_1 = \mu_2 = -1$. In this case, the characteristic exponents must be $i/2$, as

$$-1 = \cos(\pi) + i\sin(\pi) = e^{i\pi} = e^{\tau i/2}.$$

Therefore there exists a solution of the form

$$z_1(T) = e^{iT2} p_1(T) = (\cos(T/2) + i\sin(T/2))p_1(T)$$

for these characteristic numbers. Thus, there exists a 4π periodic solution in this case.

If $|z_1(\tau)| > 1$ then μ_1, and μ_2 are positive and real. The differential equation has solutions of the form

$$z_1(T) = c_1 e^{\alpha T} p_1(T) + c_2 e^{-\alpha T} p_2(T),$$

where $\alpha > 0$ and real. Hence, an unbounded solution exists in this case.

If $z_1(\tau) < -1$ then μ_1, and μ_2 are negative and real. The differential equation has solutions of the form

$$\begin{aligned} z_1(T) &= c_1 e^{(\alpha + i/2)T} p_1(T) + c_2 e^{(-\alpha + i/2)T} p_2(T) \\ &= c_1 e^{\alpha T} e^{iT/2} p_1(T) + c_2 e^{-\alpha T} e^{iT/2} p_2(T) \\ &= c_1 e^{\alpha T} q_1(T) + c_2 e^{-\alpha T} q_2(T), \end{aligned}$$

where $\alpha > 0$ and real, and $q_1(T)$, and $q_2(T)$ are 4π periodic functions. Hence, an unbounded solution exists in this case.

The cases when $|z_1(\tau)| = 1$ form the boundary between bounded and unbounded solution regimes. So if a 2π or 4π periodic solution exists at a particular point in the AB plane, that point is on the boundary between bounded and unbounded solutions.

The discussion of boundedness above only addresses whether solutions for $z(T)$ are bounded. However, $\zeta(T)$ governs the surface fluid motion, not $z(T)$. In order for surface waves to occur, an unbounded solution for $\zeta(T)$ must exist. Since surface waves are necessary for droplet ejection, an unstable solution for $\zeta(T)$ is required to be in a droplet ejecting regime.

Recall that $z(T)$ is related to $\zeta(T)$ through Equation (3.3).

$$ze^{-2\nu k^2 T/\omega} = \zeta.$$

$\zeta(T)$ is $z(T)$ times an exponential decay factor. Thus, any point on the AB plane for which no unbounded solutions exist for $z(T)$ will also have no unbounded solutions for $\zeta(T)$. Solutions for $\zeta(T)$ can be found by substituting a solution for $z(T)$ given by Equation (3.7) into Equation (3.3), yielding

$$\zeta(T) = p_i(T)e^{(\alpha_i - 2\nu k^2/\omega)T}.$$

In order for a solution to $\zeta(T)$ to be unbounded and yield an instability, α_i (the real part of the characteristic exponent) must be $> 2\nu k^2/\omega$. Let μ be the characteristic number of greater complex magnitude. Thus, by Equation (3.8)

$$|\mu| > e^{2\nu k^2 \tau/\omega} \tag{3.18}$$

must be true in order for a solution for ζ to yield instability.

3.8 Results of Stability Analysis

Equation (3.5) was solved numerically over a wide range of parameters A and B. At each point on the A-B plane, the value of $z_1(\tau)$ was calculated and the corresponding value for $|\mu|$ was found by applying Equation (3.17).

A specific value for the minimum $|\mu|$ necessary to yield instability can be found for a fluid driven at a given forcing frequency if sub-harmonic resonance is assumed, that is if $A = .25$. For water:

$$\begin{aligned}\sigma &= 72.8\ dyne/cm \\ \rho &= 1\ g/cm^3 \\ \nu &= 1.002 * 10^{-2}\ cm^2/sec\end{aligned}$$

At a driving frequency of $\omega = 1\ MHz$ and sub-harmonic resonance, the wave number of the instability, k can be calculated by substituting these values into the equation for A. This yields $k = 1513\ cm^{-1}$. Plugging this value for k into Equation (3.18) yields the minimum $|\mu|$ necessary to yield instability, which with these parameters is $|\mu| > 1.33$.

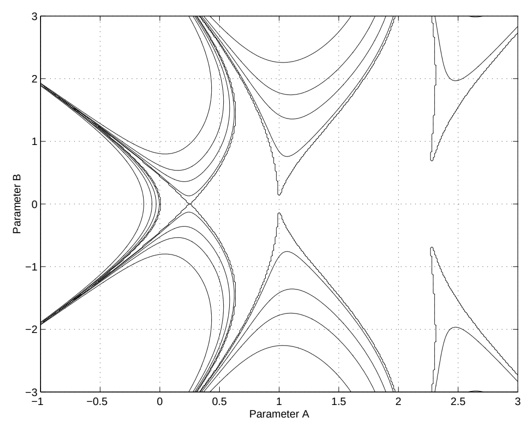

Figure 3.1: Contour plot of $|\mu|$ on the A-B plane. The values of the contours shown are $|\mu| = 1.0001, 1.5, 3, 5, 10$, with $|\mu| = 1.0001$ being the sharpest of these contours. Points inside of any of these contours have one unbounded solution for z at that value of A and B. While for ζ, one unbounded solution will exist if $|\mu|$ satisfies Equation (3.18).

For water, at sub-harmonic resonance, driven at $\omega = 1\ MHz$, $|\mu| > 1.33$ in order for a solution to yield instability. Sub harmonic resonance yields the lowest value of B that is able to support an unstable wave solution, as it dips lower than the other *tongues*. Hence, the bottom of the $|\mu| = 1.33$ sub harmonic resonance *tongue* will yield the lowest value of B that is able to generate surface wave instability.

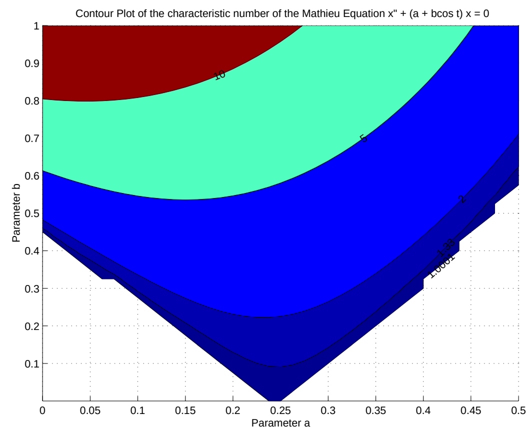

Figure 3.2: Zoom in on sub harmonic of figure (3.1).

The condition on parameter B yields a condition on the forcing acceleration, a, as $B = -ak/\omega^2$. We neglect the negative sign, as it only contributes a phase factor to the final solution, and get

$$a > 4\omega\nu k^2$$

in order for surface wave excitation to occur.

3.9 Damping Inclusion

In sections 3.7 and 3.8, a method of determining the boundary curves between surface wave producing and non wave producing regimes for a particular fluid at sub-harmonic resonance was presented. In this section, the problem of finding stability curves is approached in more general terms starting from Equation (3.4):

$$\frac{d^2z}{dT^2} + (k^3\sigma/\rho\omega^2 - 4\nu^2k^4/\omega^2 - (ak/\omega^2)\cos(T))z = 0. \tag{3.19}$$

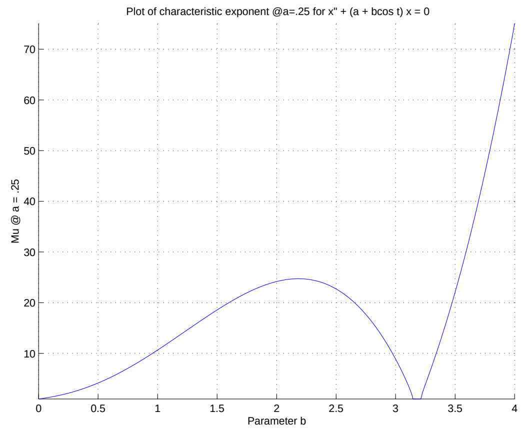

Figure 3.3: Plot of $|\mu|$ versus B at $A = .25$. The instability condition that $|\mu| > 1.33$ for water driven at 1 MHz translates into the condition that $B > .0912 \approx 4\nu k^2/\omega$. Recall that $4\nu k^2/\omega$ is the damping term from Equation (3.2), while B is the forcing amplitude. Hence, the forcing amplitude must be greater than the damping in order for a solution for ζ to be unbounded.

To simplify this equation apply the following substitutions:

$$\begin{aligned} \gamma &= 4\nu k^2/\omega \\ A_\gamma &= k^3\sigma/\rho\omega^2 \\ B_\gamma &= -(ak/\omega^2). \end{aligned}$$

This produces a form of the differential equation that still includes the coefficient from the damping term, which is separated from the other coefficients.

$$\frac{d^2z}{dT^2} + (A_\gamma - \gamma^2/4 + B_\gamma \cos(T))z = 0.$$

In terms of these coefficients, z relates to ζ, the variable governing surface wave motion, as follows:

$$ze^{-T\gamma/2} = \zeta.$$

Hence, in order for an unbounded solution in ζ to exist, the characteristic number

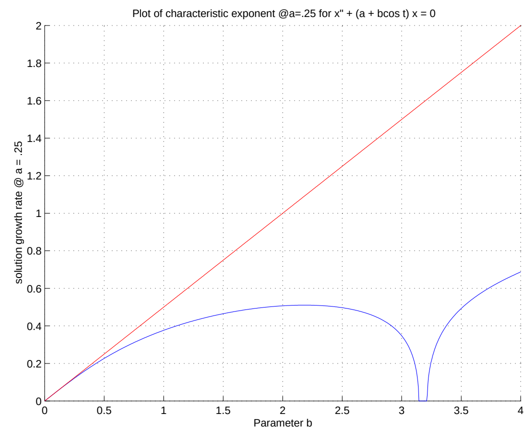

Figure 3.4: Plot of α versus B at $A = .25$. Note that $|\mu| = e^{\alpha 2\pi}$, and that $\alpha = B/2$ provides a convenient upper bound for α.

must satisfy:

$$|\mu| > e^{\gamma\tau/2}.$$

The value of the characteristic numbers were found by solving for $z_1(\tau)$ over a region of A_γ,B_γ, and γ space, and then applying Equation (3.17).

3.10 Stability Curves in ω^* vs A^* Space

In this section, the stability curves are examined in a set of non-dimensional constants. To create the differential equation that was used, begin with Equation (3.1):

$$\frac{d^2\zeta}{dt^2} + 4\nu k^2 \frac{d\zeta}{dt} + (k^3\sigma/\rho - ak\cos(\omega t))\zeta = 0. \tag{3.20}$$

Now, let K^*, A^*, and ω^* be the non-dimensional versions of the parameters k, a, and ω. In other words,

$$K^* = k/\bar{k} \tag{3.21}$$

$$A^* = a/\bar{a} \tag{3.22}$$

$$\omega^* = \omega/\bar{\omega}, \tag{3.23}$$

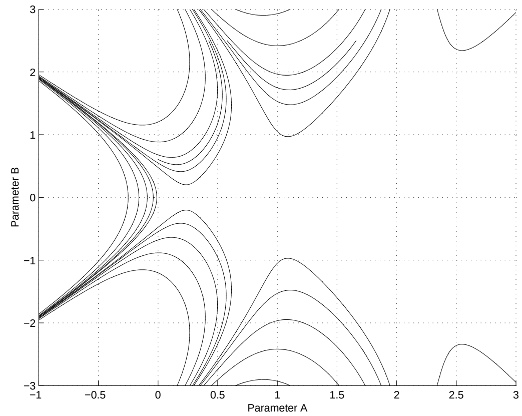

Figure 3.5: The curves plotted above are the stability boundaries in the $A_\gamma B_\gamma$ plane, for varying γ. 6 values of γ stability boundaries are plotted, with value evenly distributed on $[.2, 1]$.

where $\bar{k}$ is the characteristic inverse length scale, $\bar{a}$ is the characteristic acceleration scale, and $\bar{\omega}$ is the characteristic inverse time scale. Characteristic scale means that with the set of constants in this problem, the fluid viscosity ν and the fluid's surface tension to density ratio, σ/ρ, constants can be construct to have a particular dimension. The characteristic length scale would be the combination of these constants that gives units of length, $\nu^2/(\sigma/\rho)$. Hence,

$$\begin{aligned} \bar{k} &= (\sigma/\rho)/\nu^2 \\ \bar{a} &= (\sigma/\rho)^3/\nu^4 \\ \bar{\omega} &= (\sigma/\rho)^2/\nu^3. \end{aligned}$$

Substituting the non-dimensional version of these variables into Equation (3.1) yields:

$$\frac{d^2\zeta}{dt^2} + 4K^{*2}\frac{(\sigma/\rho)^2}{\nu^3}\frac{d\zeta}{dt} + (K^{*3}\frac{(\sigma/\rho)^6}{\nu^4} - A^*K^*\frac{(\sigma/\rho)^6}{\nu^4}\cos(\omega^*\frac{(\sigma/\rho)^2}{\nu^3}t))\zeta = 0.$$

Scaling time by $T = ((\sigma/\rho)^2/\nu^3)t$ further simplifies this differential equation:

$$\frac{d^2\zeta}{dT^2} + 4K^{*2}\frac{d\zeta}{dT} + (K^{*3} - A^*K^*\cos(\omega^*T))\zeta = 0.$$

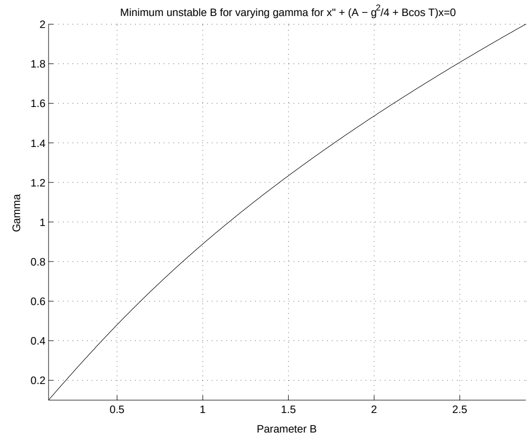

Figure 3.6: The minimum value of parameter B_γ that causes instability in the first sub harmonic resonance tongue, the tongue that approaches $A_\gamma = .25$ as γ approaches 0, was found over a range of values of γ. This is a plot of those values against the γ value at which they occur.

In terms of these variables, the standard substitution that is performed to remove the damping term is

$$ze^{-2TK^{*2}} = \zeta,$$

which simplifies the differential equation as shown below,

$$\frac{d^2z}{dT^2} + (K^{*3} - 4K^{*4} - A^*K^*\cos(\omega^*T))z = 0. \tag{3.24}$$

Using a numerical approximation, $z_1(\tau)$ was found for this differential equation over a range in the $A^*\omega^*$ plane, yielding the characteristic numbers at those points. For this equation the minimum characteristic number necessary in order for an unbounded solution to exist is:

$$|\mu| > e^{2K^{*2}\tau}.$$

For a several different K^*, the contour of this bound was plotted.

For $K^* \gg 1$,

$$12.03K^{*3} = A^* \tag{3.25}$$

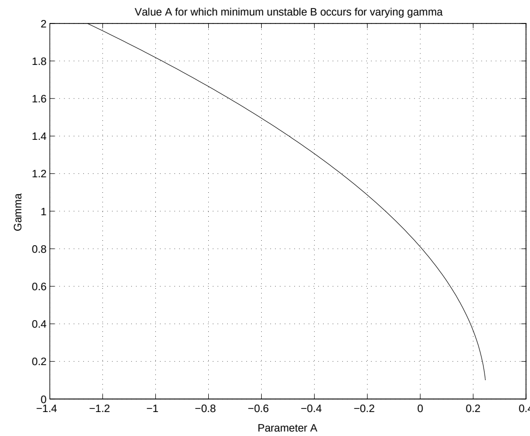

Figure 3.7: The value of A_γ at which the minimum B_γ occurs on the first sub-harmonic *tongue* is found over a range of γ. These values are plotted against the γ at which they occur in this figure.

For $K^* \gg 1$,

$$1.860 K^{*2} = \omega^* \tag{3.26}$$

Putting Equations (3.25-3.26) together yields a relationship for A^* and ω^* at large $K^* \gg 1$,

$$4.74 \omega^{*\frac{3}{2}} = A^* \tag{3.27}$$

By applying Equations (3.22) and (3.23) the dimensions on A^*, and ω^* can be recovered. Using these substitution yields:

$$4.74 \nu^{\frac{1}{2}} \omega^{\frac{3}{2}} = A \tag{3.28}$$

The acceleration required for particle ejection when $1 > \omega^* > 10^{-5}$ has been shown to be

$$1.306 \nu^{\frac{1}{2}} \omega^{\frac{3}{2}} = A$$

through experiment. [7] This scaling applies to a different range of ω^* than the scaling given in Equation (3.26) which applies when $\omega^* \gg 1$. Note the two equations agree in terms of exponents on ω and ν, but disagree on the constant in an unintuitive way.

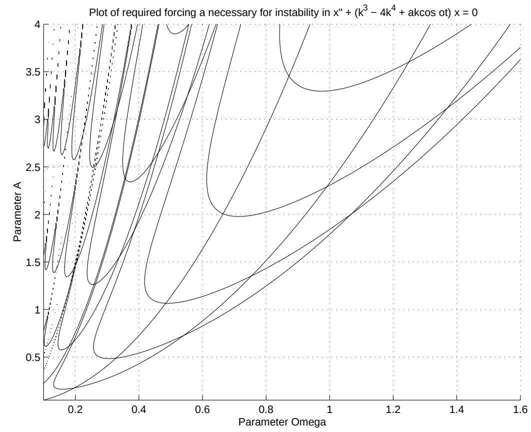

Figure 3.8: For 6 evenly spaced values of K^* ranging from .1 to .6, the boundary between regions where an unbounded solution for ζ exists and where all solutions for ζ are bounded are plotted on the $A^*\omega^*$ plane. The lowest curve corresponds to $K^* = .1$, while the highest curve corresponds to $K^* = .6$, and the other K^* values fill in the range between them.

Since wave formation is required for droplet formation, the acceleration required for droplet formation must be greater than the acceleration required for droplet ejection. The constants shown suggest otherwise, although they apply in different ranges of ω^*.

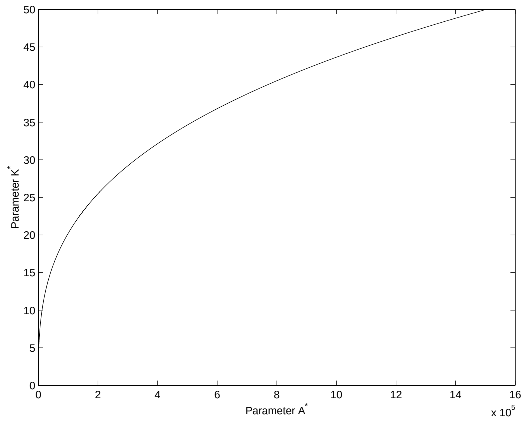

Figure 3.9: The minimum A^* required in order for an unbounded solution for ζ to exist is plotted at varying values of K^*.

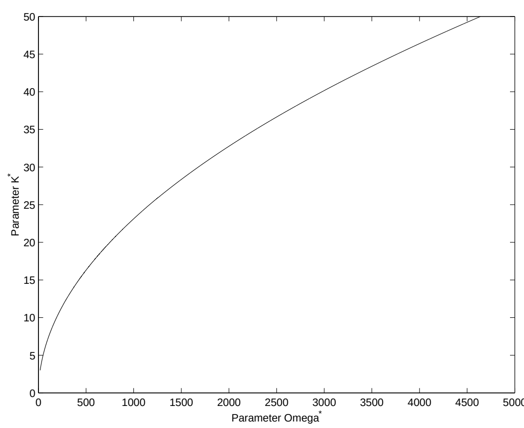

Figure 3.10: The value of ω^* at which the minimum A^* required for surface wave excitation to occur is plotted against varying values of K^*.

Chapter 4

Fourier Methods for Weakly Viscous Analysis

As it was shown in the characterization of solutions to the Mathieu Equation in Floquet form, 2π and 4π periodic solutions form the boundary between growing and bounded solutions. Finding these periodic solutions solves for the growing regions in parameter space. Hence, if a solution to Equation (3.1) can be written as a Fourier series, then this solution is periodic and part of the boundary between growing and bounded solutions. Thus, Fourier methods can be applied to solve the problem of Mathieu Equation presented in Chapter 3.

4.1 Fourier Methods

Recall that the differential equation,

$$\frac{d^2\zeta}{dt^2} + 4\nu k^2 \frac{d\zeta}{dt} + (k^3\sigma/\rho - ak\cos(\omega t))\zeta = 0,$$

can be used to model the motion of the surface of the fluid with viscosity ν, surface tension σ, and density ρ, where k is the the wave number of the surface oscillations, and the bottom of the container holding the fluid is oscillating with maximum acceleration a and frequency ω.

In section 3.2, this equation was non-dimensionalized, and then simplified by using a substitution. In this case, the constants in the equation will keep their dimensions but the substitution will still be applied. Applying the substitution,

$$ze^{-2\nu k^2 T} = \zeta,$$

yields the following differential equation:

$$\frac{d^2z}{dt^2} + (k^3\sigma/\rho - 4\nu^2k^4 - ak\cos(\omega t))z = 0.$$

To further reduce the number of constants that appear in the equation, let

$$\begin{aligned} A_o &= k^3\sigma/\rho - 4\nu^2k^4 \\ B_o &= -ak. \end{aligned}$$

This gives a simplified version of the differential equation,

$$\frac{d^2z}{dt^2} + (A_o - B_o\cos(\omega t))z = 0. \tag{4.1}$$

Note that

$$\begin{aligned} A &= A_o/\omega^2 \\ B &= B_o/\omega^2, \end{aligned}$$

where A and B are the same parameters that appear in Equation (5).

As discussed in section 3.7, on the boundary between bounded and unbounded solutions to $z(T)$ either a 2π or 4π periodic solution exists. Recall that $T = \omega t$. With unscaled time t, these solutions are either $2\pi/\omega$ or $4\pi/\omega$ periodic. If for a particular set of values for A_o, B_o, and ω, a $2\pi/\omega$ or $4\pi/\omega$ periodic solution exists to Equation (4.1), a 2π or 4π periodic solution also exists to Equation (3.5) for $A = A_o/\omega^2$ and $B = B_o/\omega^2$. Let $x(t)$ be a $2\pi/\omega$ periodic solution to Equation (4.1). This solution can be written as a Fourier series as follows:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \tag{4.2}$$

and taking two derivatives of $x(t)$ yields:

$$\frac{d^2x}{dt^2}(t) = \sum_{n=-\infty}^{\infty} -n^2\omega^2 c_n e^{in\omega t}.$$

Plugging these results into Equation (4.1) gives:

$$\begin{aligned}
0 &= \sum_{n=-\infty}^{\infty} -n^2\omega^2 c_n e^{in\omega t} + (A_o + B_o\cos(\omega t))\sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \\
&= \sum_{n=-\infty}^{\infty} -n^2\omega^2 c_n e^{in\omega t} + \sum_{n=-\infty}^{\infty} (A_o + \frac{B_o}{2}(e^{i\omega t} + e^{-i\omega t}))c_n e^{in\omega t} \\
&= \sum_{n=-\infty}^{\infty} -n^2\omega^2 c_n e^{in\omega t} + \sum_{n=-\infty}^{\infty} A_o c_n e^{in\omega t} + \frac{B_o c_n}{2}e^{i(n+1)\omega t} + \frac{B_o c_n}{2}e^{i(n-1)\omega t} \\
&= \sum_{n=-\infty}^{\infty} -n^2\omega^2 c_n e^{in\omega t} + A_o c_n e^{in\omega t} + \sum_{n=-\infty}^{\infty} \frac{B_o c_n}{2}e^{i(n+1)\omega t} + \sum_{n=-\infty}^{\infty} \frac{B_o c_n}{2}e^{i(n-1)\omega t}.
\end{aligned}$$

Re-indexing the last two sums yields:

$$\begin{aligned}
0 &= \sum_{n=-\infty}^{\infty} -n^2\omega^2 c_n e^{in\omega t} + A_o c_n e^{in\omega t} + \sum_{n=-\infty}^{\infty} \frac{B_o c_{n-1}}{2}e^{in\omega t} + \sum_{n=-\infty}^{\infty} \frac{B_o c_{n+1}}{2}e^{in\omega t} \\
&= \sum_{n=-\infty}^{\infty} -n^2\omega^2 c_n e^{in\omega t} + A_o c_n e^{in\omega t} + \frac{B_o c_{n-1}}{2}e^{in\omega t} + \frac{B_o c_{n+1}}{2}e^{in\omega t} \\
&= \sum_{n=-\infty}^{\infty} (-n^2\omega^2 c_n + A_o c_n + \frac{B_o c_{n-1}}{2} + \frac{B_o c_{n+1}}{2})e^{in\omega t}.
\end{aligned}$$

In order for this equation to hold, the coefficient on each term must equal zero. Therefore,

$$-n^2\omega^2 c_n + A_o c_n + \frac{B_o c_{n-1}}{2} + \frac{B_o c_{n+1}}{2} = 0,$$

for all $n \in \mathbb{Z}$. This can be written in matrix form as follows:

$$\begin{bmatrix} \ddots & & & & & & \\ & A_o-4\omega^2 & \frac{B_o}{2} & 0 & 0 & 0 & \\ & \frac{B_o}{2} & A_o-\omega^2 & \frac{B_o}{2} & 0 & 0 & \\ & 0 & \frac{B_o}{2} & A_o & \frac{B_o}{2} & 0 & \\ & 0 & 0 & \frac{B_o}{2} & A_o-\omega^2 & \frac{B_o}{2} & \\ & 0 & 0 & 0 & \frac{B_o}{2} & A_o-4\omega^2 & \\ & & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix}$$

$= AC = 0.$

This matrix equation can be solved by finding which values of A_o, B_o, and ω make A singular, and hence make $\det(A) = 0$. The equation can also be rearranged to a two matrix eigenvalue problem of the form $AC = \omega^2 BC$ as follows:

$$\begin{bmatrix} \ddots & & & & & & \\ & A_o & \frac{B_o}{2} & 0 & 0 & 0 & \\ & \frac{B_o}{2} & A_o & \frac{B_o}{2} & 0 & 0 & \\ & 0 & \frac{B_o}{2} & A_o & \frac{B_o}{2} & 0 & \\ & 0 & 0 & \frac{B_o}{2} & A_o & \frac{B_o}{2} & \\ & 0 & 0 & 0 & \frac{B_o}{2} & A_o & \\ & & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix} = \omega^2 \begin{bmatrix} \ddots & & & & & & \\ & -4 & 0 & 0 & 0 & 0 & \\ & 0 & -1 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & -1 & 0 & \\ & 0 & 0 & 0 & 0 & -4 & \\ & & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix}. \tag{4.3}$$

Hence, a $2\pi/\omega$ periodic solution exists for any A_o, B_o, and ω that satisfy this equation. This two matrix eigenvalue problem was solved over a region in the A_oB plane. For each point in the plane that had a positive eigenvalue, ω^2, the point $(A_o/\omega^2, B_o/\omega^2)$ was plotted on the AB plane. At these points, 2π periodic solutions to Equation(3.5) exist.

The $x(t)$ given in Equation(4.2) is $2\pi/\omega$ periodic, but solutions on the boundary of unstable solution existence can also be $4\pi/\omega$ periodic. Let $x(t)$ be $4\pi/\omega$ periodic.

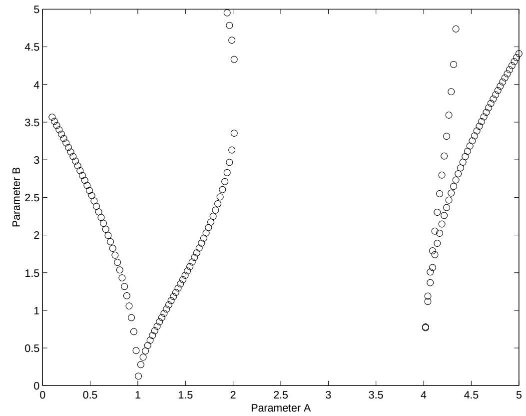

Figure 4.1: Plot of AB points that yield 2π periodic solutions to Equation (3.5). Note that these were obtained through solving the eigenvalue problem given in Equation (4.3) over a region on the A_oB plane, and finding the corresponding point on the AB plane. These tongues correspond to harmonic resonance.

Hence, $x(t)$ can as a Fourier series of the form

$$x(t) = \sum_{n=-\infty,odd}^{\infty} c_n e^{in\omega t/2}.$$

Note that for even n, the $2\pi/\omega$ periodic series is recovered, and that it is only for odd n that $e^{in\omega t/2}$ will be $4\pi/\omega$ periodic.

Writing $x(t)$ in this form, yields

$$\frac{d^2x}{dt^2}(t) = \sum_{n=-\infty,odd}^{\infty} (-n^2\omega^2 c_n/4)e^{in\omega t/2}.$$

as the second derivative of $x(t)$.

Through manipulations similar to those given for the $2\pi/\omega$ periodic Fourier series, the following two matrix eigenvalue problem is obtained:

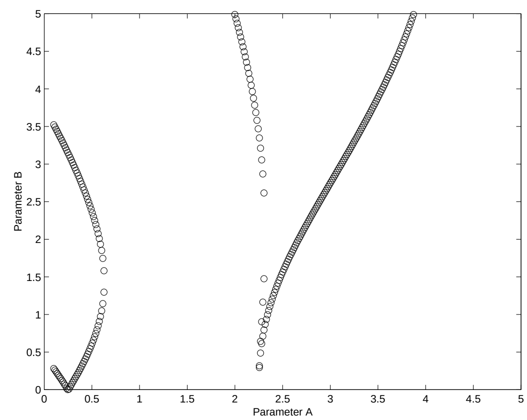

Figure 4.2: Plot of AB points that yield 4π periodic solutions to Equation (3.5). Note that these were obtained through solving the eigenvalue problem given in Equation (4.4) over a region on the A_oB plane, and finding the corresponding point on the AB plane. These tongues correspond to harmonic resonance.

$$\begin{bmatrix} \ddots & & & & & \\ & A_o & \frac{B_o}{2} & 0 & 0 & \\ & \frac{B_o}{2} & A_o & \frac{B_o}{2} & 0 & \\ & 0 & \frac{B_o}{2} & A_o & \frac{B_o}{2} & \\ & 0 & 0 & \frac{B_o}{2} & A_o & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ c_{-3} \\ c_{-1} \\ c_1 \\ c_3 \\ \vdots \end{bmatrix} = \frac{\omega^2}{4} \begin{bmatrix} \ddots & & & & & \\ & -9 & 0 & 0 & 0 & \\ & 0 & -1 & 0 & 0 & \\ & 0 & 0 & -1 & 0 & \\ & 0 & 0 & 0 & -9 & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ c_{-3} \\ c_{-1} \\ c_1 \\ c_3 \\ \vdots \end{bmatrix}. \tag{4.4}$$

Translating the A_o, B_o, and ω solutions to this eigenvalue problem into the AB plane gives 4π periodic solutions to Equation (3.5).

4.2 *Fourier Method for Finding Stability Curves in the $A^*\omega^*$ Plane*

In this section, the $K^* \gg 1$ scalings from section 3.10 are reproduced using Fourier methods. The discussion in this section starts from the differential equation given by

Equation (3.24):

$$\frac{d^2\zeta}{dT^2} + 4K^{*2}\frac{d\zeta}{dT} + (K^{*3} - A^*K^*\cos(\omega^*T))\zeta = 0.$$

In order to establish that a periodic solution will exist only on the boundary between regions where an unbounded solution for ζ exists, and where all solutions are bounded, the first step in to put the differential equation into matrix form so that Floquet Theory results can be applied. The differential equation above when written as a system of first order differential equations is:

$$\begin{bmatrix} \frac{d\zeta_1}{dT} \\ \frac{d\zeta_2}{dT} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K^{*3} + A^*K^*\cos(\omega^*T) & -4K^{*2} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}.$$

Note that the trace of the coefficient matrix is non zero here,

$$tr(\begin{bmatrix} 0 & 1 \\ -K^{*3} + A^*K^*\cos(\omega^*T) & -4K^{*2} \end{bmatrix}) = -4K^{*2}.$$

From Floquet Theory, we can conclude that the product of the characteristic numbers of the system,

$$\mu_1\mu_2 = e^{\int 2\pi/\omega^* 0 - 4K^{*2}dS} = e^{-8\pi K^{*2}/\omega^*} < 1,$$

since $\omega^* > 0$. Hence if one of the characteristic numbers, say μ_1, $= 1$ then $\mu_2 = e^{-8\pi K^{*2}/\omega^*} < 1$. Thus, if one of the characteristic numbers is 1, then that point in parameter space is on the boundary of surface excitation. Similarly, if one of the characteristic numbers is -1, that point in parameter space must be on the boundary. This implies that if a $2\pi/\omega^*$ or $4\pi/\omega^*$ periodic solution exists, the values of A^*, ω^*, and K^* are on the surface wave excitation boundary.

The next task is to actually find such periodic solutions. Let $\zeta(t)$ be a $2\pi/\omega^*$ periodic solution to the differential equation. Hence it can be written in as a Fourier series,

$$\zeta(t) = \sum_{n=-\infty}^{\infty} (c_n + ib_n)e^{in\omega^* t}. \tag{4.5}$$

Note that this series must have complex coefficients. Otherwise, differentiating once would not be a closed operation for the series. Also, note that in order for this series to give a real function, the following must hold true

$$c_n = c_{-n},$$

and

$$b_n = -b_{-n}.$$

In order to see this, lets examine Equation (4.5) a bit more extensively. Breaking down $e^{in\omega^* t}$ into sin and cos functions gives,

$$\begin{aligned}
\zeta(t) &= \sum_{n=-\infty}^{\infty} (c_n + ib_n)e^{in\omega^* t} \\
&= \sum_{n=-\infty}^{\infty} (c_n + ib_n)(\cos(n\omega^* t) + i\sin(n\omega^* t)) \\
&= c_0 + \\
ib_0 & \textstyle\sum \;{}_{n=1}^{\infty}(c_n + ib_n)(\cos(n\omega^* t) + i\sin(n\omega^* t)) + (c_{-n} + b_{-n})(\cos(-n\omega^* t) + i\sin(-n\omega^* t)) \\
&= c_0 + ib_0 \sum_{n=1}^{\infty}(c_n + ib_n)(\cos(n\omega^* t) + i\sin(n\omega^* t)) + (c_{-n} + b_{-n})(\cos(n\omega^* t) - i\sin(n\omega^* t)) \\
&= c_0 + ib_0 \sum_{n=1}^{\infty}(c_n + c_{-n} + i(b_n + b_{-n}))(\cos(n\omega^* t) + (-b_n + b_{-n} + i(c_n - c_{-n}))\sin(n\omega^* t)).
\end{aligned}$$

Thus, in order for ζ to be a real function, $c_n = c_{-n}$, $b_n = -b_{-n}$, and $b_o = 0$.

Differentiating to find the other terms in the differential equation, yields:

$$\frac{d\zeta}{dt}(t) = \sum_{n=-\infty}^{\infty} n\omega^*(-b_n + ic_n)e^{in\omega^* t},$$

and

$$\frac{d^2\zeta}{dt^2}(t) = \sum_{n=-\infty}^{\infty} -n^2\omega^{*2}(c_n + ib_n)e^{in\omega^* t}.$$

Plugging these results into Equation (3.24) gives:

$$\begin{aligned}
0 &= \sum_{n=-\infty}^{\infty} -n^2\omega^{*2}(c_n + ib_n)e^{in\omega^* t} + (4K^{*2}) \sum_{n=-\infty}^{\infty} n\omega^*(-b_n + ic_n)e^{in\omega^* t} \\
&+ (K^{*3} + A^* K^* \cos(\omega^* t)) \sum_{n=-\infty}^{\infty} (c_n + ib_n)e^{in\omega^* t} \\
&= \sum_{n=-\infty}^{\infty} [-n^2\omega^{*2}c_n - 4K^{*2}n\omega^* b_n + K^{*3}c_n + i(-n^2\omega^{*2}b_n + 4K^{*2}n\omega^* c_n + K^{*3}b_n)]e^{in\omega^* t} \\
&+ \sum_{n=-\infty}^{\infty} (1/2)A^* K^*(e^{i\omega^* t} + e^{-i\omega^* t})(c_n + ib_n)e^{in\omega^* t} \\
&= \sum_{n=-\infty}^{\infty} [-n^2\omega^{*2}c_n - 4K^{*2}n\omega^* b_n + K^{*3}c_n + i(-n^2\omega^{*2}b_n + 4K^{*2}n\omega^* c_n + K^{*3}b_n)]e^{in\omega^* t} \\
&+ \sum_{n=-\infty}^{\infty} \frac{A^* K^*}{2}(c_n + ib_n)e^{i(n+1)\omega^* t} + \sum_{n=-\infty}^{\infty} \frac{A^* K^*}{2}(c_n + ib_n)e^{i(n-1)\omega^* t}
\end{aligned}$$

Re-indexing the last two sums yields:

$$\begin{aligned}
0 &= \sum_{n=-\infty}^{\infty} [-n^2\omega^{*2}c_n - 4K^{*2}n\omega^* b_n + K^{*3}c_n + i(-n^2\omega^{*2}b_n + 4K^{*2}n\omega^* c_n + K^{*3}b_n)]e^{in\omega^* t} \\
&+ \sum_{n=-\infty}^{\infty} \frac{A^* K^*}{2}(c_{n-1} + ib_{n-1})e^{in\omega^* t} + \sum_{n=-\infty}^{\infty} \frac{A^* K^*}{2}(c_{n+1} + ib_{n+1})e^{in\omega^* t} \\
&= \sum_{n=-\infty}^{\infty} [(-n^2\omega^{*2}c_n - 4K^{*2}n\omega^* b_n + K^{*3}c_n + \frac{A^* K^*}{2}c_{n-1} + \frac{A^* K^*}{2}c_{n+1}) \\
&+ i(-n^2\omega^{*2}b_n + 4K^{*2}n\omega^* c_n + K^{*3}b_n + \frac{A^* K^*}{2}b_{n-1} + \frac{A^* K^*}{2}b_{n+1})]e^{in\omega^* t}
\end{aligned}$$

In order for this equation to hold, the coefficient on both the real and imaginary parts of each term in the series must equal zero. Therefore,

$$-n^2\omega^{*2}c_n - 4K^{*2}n\omega^* b_n + K^{*3}c_n + \frac{A^* K^*}{2}c_{n-1} + \frac{A^* K^*}{2}c_{n+1} = 0,$$

and

$$-n^2\omega^{*2}b_n + 4K^{*2}n\omega^* c_n + K^{*3}b_n + \frac{A^* K^*}{2}b_{n-1} + \frac{A^* K^*}{2}b_{n+1} = 0.$$

for all $n \in \mathbb{Z}$. Because of the restriction on c_{-n}, and b_{-n} that forces ζ to be a real function implies, it is only necessary to examine non negative n values. Also note that,

$$\begin{aligned} 0 &= (K^{*3} - 0^2\omega^{*2})c_0 - 4K^{*2}0\omega^* b_0 + \frac{A^*K^*}{2}c_{-1} + \frac{A^*K^*}{2}c_1 \\ &= K^{*3}c_0 + A^*K^*c_1, \end{aligned}$$

and similarly

$$\begin{aligned} 0 &= (K^{*3} - 0^2\omega^{*2})b_0 + 4K^{*2}0\omega^* c_0 + \frac{A^*K^*}{2}b_{-1} + \frac{A^*K^*}{2}b_1 \\ &= K^{*3}b_0. \end{aligned}$$

The entire system of equations can be written in matrix form as follows:

$$\begin{bmatrix} K^{*3} & 0 & A^*K^* & 0 & 0 & 0 & \\ 0 & K^{*3} & 0 & 0 & 0 & 0 & \\ \frac{A^*K^*}{2} & 0 & K^{*3} - \omega^{*2} & -4K^{*2}\omega^* & \frac{A^*K^*}{2} & 0 & \\ 0 & \frac{A^*K^*}{2} & 4K^{*2}\omega^* & K^{*3} - \omega^{*2} & 0 & \frac{A^*K^*}{2} & \\ 0 & 0 & \frac{A^*K^*}{2} & 0 & K^{*3} - 4\omega^{*2} & -8K^{*2}\omega^* & \\ 0 & 0 & 0 & \frac{A^*K^*}{2} & 8K^{*2}\omega^* & K^{*3} - 4\omega^{*2} & \\ & & & & & & \ddots \end{bmatrix} \begin{bmatrix} c_0 \\ b_0 \\ c_1 \\ b_1 \\ c_2 \\ b_2 \\ \vdots \end{bmatrix} = AC = 0. \tag{4.6}$$

This matrix equation can be solved by finding which values of K^*, A^*, and ω^* make A singular, and hence make $\det(A) = 0$. At these points, $2\pi/\omega^*$ periodic solutions to Equation(3.24) exist.

Thus far, only $2\pi/\omega^*$ periodic functions have been found, while the existence of a $4\pi/\omega^*$ function also implies boundary between excitation regimes. Let ζ be a $4\pi/\omega^*$ function, and in a similar fashion to finding the harmonic curve for Equation (3.24), ζ can be written as the following $4\pi/\omega^*$ periodic Fourier series:

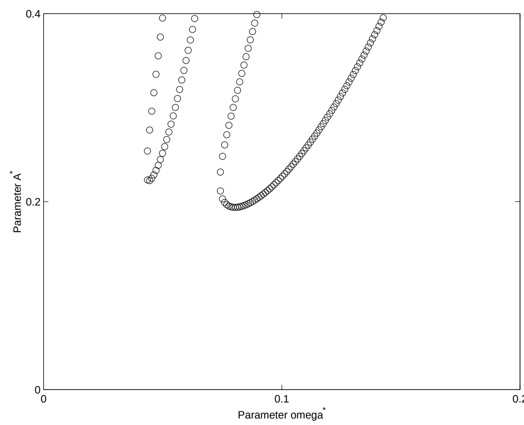

Figure 4.3: This is a plot of the ordered pairs (ω^*, A^*) that yield a non-trivial solution for Equation (4.6) with $K^* = .15$.

$$\zeta(t) = \sum_{n=-\infty,odd}^{\infty} (c_n + ib_n)e^{in\omega^* t/2}.$$

Differentiating this series yields:

$$\frac{d\zeta}{dt} = \sum_{n=-\infty,odd}^{\infty} \frac{n\omega^*}{2}(-b_n + ic_n)e^{in\omega^* t/2},$$

and

$$\frac{d^2\zeta}{dt^2} = \sum_{n=-\infty,odd}^{\infty} -\frac{n^2\omega^{*2}}{4}(c_n + ib_n)e^{in\omega^* t/2}.$$

Theses series' when plugged into Equation (3.24), and after applying similar manipulations as those used in the harmonic case, yield the following matrix equation:

$$\begin{bmatrix} K^{*3} - \frac{\omega^{*2}}{4} + \frac{A^*K^*}{2} & -2K^{*2}\omega^* & \frac{A^*K^*}{2} & 0 & \\ 2K^{*2}\omega^* & K^{*3} - \frac{\omega^{*2}}{4} - \frac{A^*K^*}{2} & 0 & \frac{A^*K^*}{2} & \\ \frac{A^*K^*}{2} & 0 & K^{*3} - \frac{9\omega^{*2}}{4} & -6K^{*2}\omega^* & \\ 0 & \frac{A^*K^*}{2} & 6K^{*2}\omega^* & K^{*3} - \frac{9\omega^{*2}}{4} & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} c_1 \\ b_1 \\ c_3 \\ b_3 \\ \vdots \end{bmatrix} = AC = 0. \tag{4.7}$$

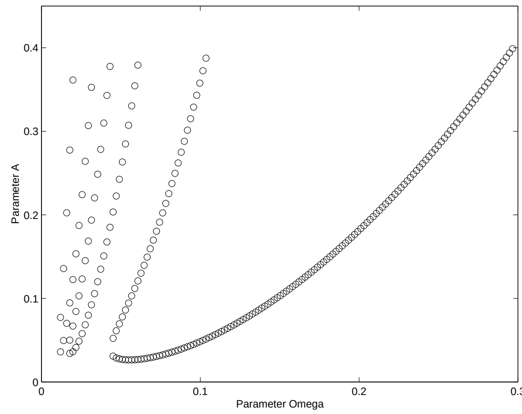

Figure 4.4: Each (ω^*, A^*) ordered pair plotted above solves the matrix Equation (4.7) with $K^* = .15$, making the curves shown the boundary for sub harmonic resonance between surface wave excitation regimes.

For $K^* \gg 1$ the relationship between A^* and K^* is

$$12.03K^{*3} = A^* \tag{4.8}$$

For $K^* \gg 1$,

$$1.860K^{*2} = \omega^* \tag{4.9}$$

Putting Equations (4.8-4.9) together yields a relationship for A^* and ω^* at large $K^* \gg 1$,

$$4.74\omega^{*\frac{3}{2}} = A^*$$

Note, that this is in agreement with Equation(3.27) and the scaling found without using Fourier methods.

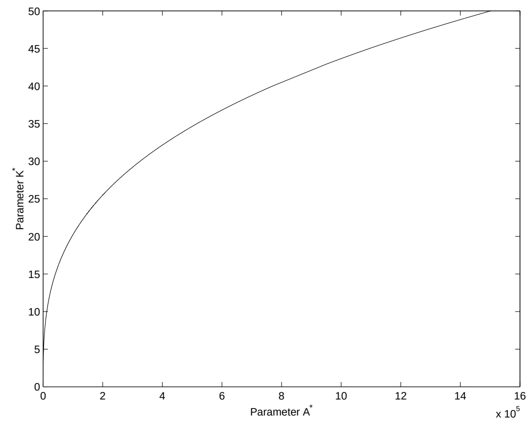

Figure 4.5: For each K^* the minimum value of A^* necessary to cause instability for either subharmonic or harmonic tongues was found. For example, for $K^* = .15$ this corresponds to point with the lowest A^* plotted in Figs. (4.3) and (4.4).

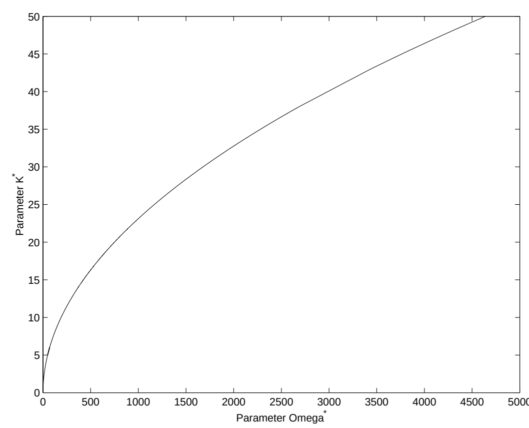

Figure 4.6: The value of ω^* at which the minimum A^* required for surface wave excitation to occur is plotted against varying values of K^*.

Chapter 5

Derivation of Hydrodynamic System

In order to deal with a highly viscous fluid, it is necessary to consider the full hydrodynamic system, which consists of the Navier Stokes Equation, the condition of fluid incompressibility, and appropriate boundary conditions including the kinematic condition. This system of partial differential equations is derived in this chapter by considering conservation laws, and Newton's second law on the surface of the fluid. Additionally, this chapter deals with the linearization of this system about the hydrostatic case. This linearization is necessary to make the system tractable.

5.1 Governing Equations in Body of Fluid

In this section, the partial differential equation governing the fluid's motion is derived from first principles. The derivation of the incompressibility condition and the stress tensor are adapted from versions that appear in *Viscous Flows* by Ockendon. [14]

Consider a fluid with free fluid surface $h(x, y, t)$, which when the fluid is at rest $= 0$, where the fluid rests in the region $z < 0$. Let $\vec{u}(x, y, z, t)$ be the vector velocity, and $P(x, y, z, t)$ be the pressure per area at all points within the fluid. Let $\rho(x, y, z, t)$ be the fluid density.

Conservation of Mass:

Consider an arbitrary piece of fluid volume, V. For a constant density, incompressible, fluid the total mass contained in that volume would simply be $M = \rho V$. Similarly in the case of variable density,

$$\int\int\int_V \rho dV = M.$$

Hence, the change in the mass in that volume over time can be written:

$$\frac{dM}{dt} = \frac{d}{dt}\int\int\int_V \rho dV = \int\int\int_V \frac{d\rho}{dt}dV. \tag{5.1}$$

The time rate of change of the mass in volume V can also be written from the perspective of density flux in and out of the volume. Since mass is conserved, if the mass in the volume decreases, this means that mass has moved out of the volume and that there was a net outward mass flux from the volume. Hence, the outward flux of mass $= -\frac{dM}{dt}$.

To get an idea of the form that the mass flux takes, consider an infinitesimal piece of surface area for our volume, dS. For this surface area, both $\vec{u}$ and ρ will be constant. Let $\hat{n}$ be the unit normal vector for this infinitesimal piece of surface area, and note that $\vec{u} \cdot \hat{n}$ is the component of velocity normal to the surface. Hence, $dS(\vec{u} \cdot \hat{n})$ is the volume flux out of dS, the rate of fluid volume leaving dS. Thus, the rate of mass leaving dS is $\rho dS(\vec{u} \cdot \hat{n})$. This a small piece of mass flux, d Mass Flux. Putting it together,

$$d(\text{Mass Flux}) = \rho(\tilde{\text{u}} \cdot \hat{\text{n}})\text{dS}.$$

Further, the total mass flux is the sum of these infinitesimal pieces of flux,

$$\text{Mass Flux} = \int\int_{\text{S}} \rho\tilde{\text{u}} \cdot \hat{\text{n}}\text{dS} = \int\int\int_{\text{V}} \nabla \cdot (\rho\tilde{\text{u}})\text{dV}.$$

Note observe that the application of the divergence theorem in converting from a surface integral to a volume integral. Hence,

$$\int\int\int_V \frac{d\rho}{dt}dV = -\int\int\int_V \nabla \cdot (\rho\vec{u})dV,$$

and since these volumes are arbitrary, the integrands must be equivalent,

$$\frac{d\rho}{dt} + \nabla \cdot (\rho \vec{u}) = 0. \tag{5.2}$$

In the case of an incompressible fluid, the density of the fluid is constant. This makes sense since no matter how strongly pressure squeezes an incompressible, the fluid would not change density. When assuming an incompressible fluid, Equation (5.2) reduces to

$$\nabla \cdot \vec{u} = 0, \tag{5.3}$$

since $\frac{d\rho}{dt} = 0$.

Conservation of Linear Momentum: Let $\vec{W}$ be the total momentum in an arbitrary volume, V. Consider an infinitesimal volume of fluid, dV, that is small enough that $\vec{u}$ and ρ are constant over dV. The infinitesimal linear momentum in that piece of fluid is

$$d\vec{W} = \rho dV \vec{u},$$

by applying the basic mechanics equation, momentum = mass*velocity. Summing over these infinitesimal pieces of fluid yields

$$\vec{W} = \int\int\int_V \rho \vec{u} dV$$

Further, the time rate of change of the momentum can be written:

$$\frac{d\vec{W}}{dt} = \frac{d}{dT} \int\int\int_V \rho \vec{u} dV = \int\int\int_V \frac{d(\rho \vec{u})}{dt} dV.$$

A natural question to ask is, what would cause the momentum in a piece of fluid volume V to change? Momentum can change by the application of forces to the fluid volume. Additionally, if particles with high momentum leave and are replaced by particles of smaller momentum, there is an outward flux of momentum from the volume. The momentum flux is another cause of momentum change in the volume.

Lets take a look at forces that could be acting on the fluid volume. These break down into two categories, body forces and surface forces. The distinction here is that body forces act inside of the volume under consideration, while surface forces act only on the surface of the volume. Let $\vec{F}$ be the force per unit mass of body forces applied to an infinitesimal piece of fluid. Hence,

$$\text{Net Body Forces} = \int\int\int_{\mathrm{V}} \rho\tilde{\mathrm{F}}\mathrm{dV}.$$

As for forces applied on the surface of the fluid, there are two such forces to consider, the pressure exerted on the volume, and the viscosity exerted on the volume. Lets take a look at the pressure exerted on the fluid volume. First note that the pressure on the surface of the volume is the only pressure that will effect the volume. In other words, pressure at points inside of the volume do not cause any change in the net momentum of the volume. To see this imagine splitting the volume into smaller volumes that fill the larger volume completely, so that the boundaries of the small volumes are in complete contact leaving no empty space. The pressure exerted on a face of one of these internal pieces of volume will be balanced by the pressure on the face that it is in contact with. Hence, the two forces counteract leaving no net force, and no net change in momentum. Hence the net force exerted by the pressure on the volume is due only to pressure on the surface of the volume. Thus the net pressure can be written as follows:

$$\text{Net Pressure Force} = \int\int_{\mathrm{S}} \mathrm{P}\hat{\mathrm{n}}\mathrm{dS} = -\int\int\int_{\mathrm{V}} \nabla\mathrm{P}\mathrm{dV},$$

where the divergence theorem was used the last step, and the negative sign shows that pressure exerts an inward force.

Now consider the force exerted by viscosity on an infinitesimal piece of fluid volume, dV. Let this piece of volume have horizontal velocity $\vec{v}$, and let z be the vertical position coordinate. The fluid just above and just below the infinitesimal piece act on

dV. Since the fluid is assumed to be Newtonian, the magnitude of the force exerted by the piece of fluid just above dV varies linearly with the magnitude of the vector difference of the velocity of the piece of dV and the velocity of dV. The direction of the force is in the same direction as this same difference vector. Hence, if the fluid is moving at a uniform velocity, no viscous forces are experienced. Additionally, if $\frac{\partial^2 \vec{v}}{dz^2} = 0$ at dV then the force exerted by the piece of fluid above dV is canceled by the force exerted by the piece just below dV, leaving no net force. In fact, the magnitude of the force in the horizontal direction caused by the vertically translated into fluid pieces is proportional to this second derivative. Similarly, the magnitude of the force in the horizontal direction caused by the horizontally translated fluid is proportional to $\frac{\partial^2 \vec{v}}{dx^2}$. More generally, the net force of viscosity on an infinitesimal piece of volume dV is proportional to $(\frac{\partial^2 \vec{v}}{dx^2} + \frac{\partial^2 \vec{v}}{dz^2})\hat{x} + (\frac{\partial^2 \vec{w}}{dx^2} + \frac{\partial^2 \vec{w}}{dz^2})\hat{z}$, where $\vec{w}$ is the vertical component of velocity, and the $\hat{z}$ term is the analog of the $\hat{x}$ term when vertical velocities are considered instead of horizontal. Let ν be the proportionality constant, known as the kinematic viscosity of the fluid, and note that for a two dimensional fluid:

$$\begin{aligned} \text{Force of Viscosity on dV} &= \nu \left[(\frac{\partial^2 \vec{v}}{dx^2} + \frac{\partial^2 \vec{v}}{dz^2})\hat{x} + (\frac{\partial^2 \vec{w}}{dx^2} + \frac{\partial^2 \vec{w}}{dz^2})\hat{z} \right] \\ &= \nu \nabla^2 \vec{u}. \end{aligned}$$

Hence, the net force of viscosity over a piece of volume, V is:

$$\text{Net Viscous Force} = \int\int\int_{\text{V}} \rho \nabla^2 \tilde{\text{u}} \text{dV}.$$

Lets now consider momentum flux. Considering an infinitesimal piece of surface area, dS, the momentum flux is analogous to the mass flux.

$$d\overrightarrow{\text{Momentum}} \text{ Flux} = (\rho\vec{u})(\vec{u} \cdot \hat{n})dS$$

Similarly, the total momentum flux is the sum of the flux over these infinitesimal pieces of surface area,

$$\vec{\text{Momentum}}\text{ Flux} = \int\int_S (\rho\vec{u})(\vec{u}\cdot\hat{n})dS = \int\int\int_V \nabla\cdot((\rho\vec{u})\vec{u})dV$$

Note that another way to write the integrand of that final volume integral, which clarifies the action of the ∇ operator, is

$$\nabla\cdot((\rho\vec{u})\vec{u}) = \nabla\cdot(\rho u_x\vec{u})\hat{x} + \nabla\cdot(\rho u_y\vec{u})\hat{y} + \nabla\cdot(\rho u_z\vec{u})\hat{z}$$

In index notation, this is equivalent to $\nabla\cdot(\rho u_i\vec{u})$.

Now that the we have described the various ways that the linear momentum in our volume can change, it is time to put these ideas together. First recall the mechanics equation $\frac{dMomentum}{dt}$ = Net Force, where in the case of fluid in a particular volume, the net force is made up of body and surface forces. Secondly, notice that a positive outward flux of momentum will reduce the the momentum. This suggests that:

$$\frac{d\vec{W}}{dt} = \text{Net Body Forces} + \text{Net Surface Forces} - \text{Momentum Flux.}$$

Substituting the previously derived expressions,

$$\begin{aligned}\int\int\int_V \frac{d(\rho\vec{u})}{dt}dV &= \int\int\int_V \rho\vec{F}dV - \int\int\int_V \nabla P dV + \int\int\int_V \rho\nu\nabla^2\vec{u}dV \\ &- \int\int\int_V \nabla\cdot((\rho\vec{u})\vec{u})dV.\end{aligned}$$

Note that all of these integrals are over the same arbitrary volume, V. Hence, the integrands are equivalent. Thus,

$$\frac{d(\rho\vec{u})}{dt} = \rho\vec{F} - \nabla P + \rho\nu\nabla^2\vec{u} - \nabla\cdot((\rho\vec{u})\vec{u}). \tag{5.4}$$

Lets consider Equation (5.4) under the assumption of incompressiblity. Hence ρ is a constant, and $\nabla\cdot\vec{u} = 0$ by Equation (5.2). Under this assumption, the term for momentum flux can be significantly simplified.

$$
\begin{aligned}
\nabla \cdot ((\rho u_i)\vec{u}) &= \rho \nabla (u_i \vec{u}) \\
&= \rho \frac{d}{dx_j}(u_i u_j) \\
&= \rho (\frac{du_i}{dx_j} u_j + u_i \frac{du_j}{dx_j}) \\
&= \rho (\vec{u} \cdot (\nabla u_i) + u_i (\nabla \cdot \vec{u})) \\
&= \rho ((\vec{u} \cdot \nabla) u_i) + u_i (\nabla \cdot \vec{u})) \\
&= \rho ((\vec{u} \cdot \nabla) \vec{u}) + \vec{u} (\nabla \cdot \vec{u})) \\
&= \rho ((\vec{u} \cdot \nabla) \vec{u})
\end{aligned}
$$

where in the last line, Equation (5.2) has been applied. This simplifies Equation (5.4) to:

$$\frac{d\vec{u}}{dt} + (\vec{u} \cdot \nabla)\vec{u}) = \vec{F} - (1/\rho)\nabla P + \nu \nabla^2 \vec{u}. \tag{5.5}$$

Equations (5.2) and (5.5) are the governing equations for fluid motion in the body of the fluid.

5.2 Governing Equations for Surface of Fluid

The first observation to make about the fluid surface is that it moves with the body of the fluid. Point being, the rate of change of the position of the surface, $h(x, y, t)$ is equal to velocity of the z coordinate at the surface, u_z. Hence,

$$u_z = \frac{dh(x,t)}{dt} = \frac{\partial h}{\partial t} + \frac{dx}{dt}\frac{\partial h}{\partial x} = \frac{\partial h}{\partial t} + u_x \frac{\partial h}{\partial x} \tag{5.6}$$

This equation is known as the kinematic condition.

Consider a two dimensional infinitesimal fluid box. Let σ_{ij} be the force per unit area exerted in the $\hat{i}$ direction acting on the side of the box with outward normal vector $\hat{j}$. σ_{ij} is known as the stress tensor. The two types of force that play a role in the tensor are pressure and viscous forces. For pressure, a force proportional to the fluid's pressure at that point will push inward on all sides of the box. Hence, $-p\delta_{ij}$ is the pressure's contribution to the stress tensor.

As discussed previously, the force of viscosity in the direction of $\hat{i}$ on the side with normal $\hat{j}$ is proportional to $\frac{\partial \vec{u}_i}{\partial x_j}$, as the fluid is assumed to be Newtonian. Hence, the viscosity contributes $\nu \frac{\partial \vec{u}_i}{\partial x_j}$ to σ_{ij}.

Now consider the angular momentum of the box, by first giving the box side lengths δ_1 and δ_2. The net angular momentum of the center of the box is:

$$\text{Net Angular Momentum} = 2(\sigma_{12}\delta_2)\delta_1/2 - 2(\sigma_{12}\delta_1)\delta_2/2.$$

If the rectangle is shrunk to zero, the angular momentum of the point must be zero in order to keep the angular momentum in the fluid finite. Thus, $\sigma_{12} = \sigma_{21}$, implying that σ_{12} must have a $\frac{\partial u_2}{\partial x_1}$ in the contribution of viscosity to the stress tensor, and a similar statement can be made regarding σ_{21}. Thus,

$$\sigma_{ij} = -p\delta_{ij} + \nu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}). \tag{5.7}$$

Note that the viscosity's contribution to the pressure has been neglected as it is a smaller order effect than the pressure and tangential viscosity.

Both tangential and normal forces balance on the surface of the fluid. In addition to the forces that act in the body of the fluid, surface tension also plays a role on the surface for the fluid. Let $\hat{n}$ be the normal vector at the fluid surface and $\hat{t}$ be the surface tangent vector. Surface tension acts normal to the surface of the fluid and attempts to reduce the surface area of the fluid. Point being, that the surface tension tries to reduce the absolute value of the curvature. Hence when curvature is positive, the surface is concave up, the force exerted by surface tension acts parallel to $\hat{n}$, while if the curvature is negative, the force exerted by surface tension acts anti parallel to $\hat{n}$. So the force of surface tension can be written:

$$\text{Force of Surface Tension} = T\kappa\hat{n}$$

where κ is the curvature of the surface, and $T = (\sigma/\rho)$ is a constant. Since the force exerted in the $\hat{n}$ direction on the $\hat{n}$ face of a piece of fluid can be written as $\hat{n}_i \cdot \sigma_{ij} \cdot \hat{n}_j$, the force balance equation for the normal direction on the fluid surface can be written as

$$\hat{n}_i \cdot \sigma_{ij} \cdot \hat{n}_j + T\kappa = 0. \tag{5.8}$$

As for the tangential force balance, there are no additional forces present on the fluid surface that are not present in the body of the fluid. Therefore,

$$\hat{t}_i \sigma_{ij} \hat{n}_j = 0 \tag{5.9}$$

These force balance equations when joined with,

$$\frac{d\vec{u}}{dt} + (\vec{u} \cdot \nabla)\vec{u}) = \vec{F} - (1/\rho)\nabla P + \nu\nabla^2\vec{u}, \tag{5.10}$$

$$u_z = \frac{\partial h}{\partial t} + u_x \frac{\partial h}{\partial x} + u_y \frac{\partial h}{\partial y}, \tag{5.11}$$

and

$$\nabla \cdot \vec{u} = 0, \tag{5.12}$$

fully describe the motion of the fluid. This set of equations is the full hydrodynamic system.

5.3 Hydrostatics and Linearizing the System

In the hydrostatic situation, $\vec{u}(x, y, z, t) = 0$ and $h(x, y, t) = 0$. In this situation, there is only one relevant body force, gravity, and thus $\vec{F} = -g\hat{z}$. Applying Equation (5.5) with these conditions yields:

$$\vec{F} = (1/\rho)\nabla \bar{P}$$

and thus

$$\nabla \bar{P} = -\rho g \hat{z},$$

where $\bar{P}$ is the order zero component of the pressure. This gives us three equations on the partial derivatives of $\bar{P}$, namely:

$$\begin{aligned} \frac{\partial \bar{P}}{\partial x} &= 0 \\ \frac{\partial \bar{P}}{\partial y} &= 0 \\ \frac{\partial \bar{P}}{\partial z} &= -\rho g, \end{aligned}$$

which imply that for the hydrostatic case,

$$P = P_o - \rho g z \tag{5.13}$$

where P_o is the pressure at $z = 0$, the atmospheric pressure.

As a means of simplifying the full hydrodynamic system, assume that the system is near the hydrostatic situation. Making this assumption allows the system to linearized about the hydrostatic case. In other words, deviations from the hydrostatic case are treated as order ϵ effects. Hence, the relevant variables in the problem can be written as

$$\vec{u}(x,z,t) = 0 + \epsilon\hat{u}(x,z,t) + \ldots \tag{5.14}$$

$$h(x,t) = 0 + \epsilon\hat{h}(x,t) + \ldots \tag{5.15}$$

$$P(x,z,t) = P_o - \rho g z + \epsilon\hat{P}(x,z,t) + \ldots \tag{5.16}$$

$$\hat{n}_i \approx z \tag{5.17}$$

$$\hat{t}_i \approx x \tag{5.18}$$

$$\kappa \approx \nabla^2\hat{h}, \tag{5.19}$$

for the 2 dimensional problem. The fluid velocity at the surface can be written as:

$$\vec{u}(x,h,t) = \epsilon\hat{u}(x,o,t) + \cdots = \epsilon(\hat{v}(x,0,t), 0, \hat{w}(x,0,t)) + \ldots, \tag{5.20}$$

where in the last step, the velocity field has been broken down into components. The linearized stress tensor is,

$$\sigma_{ij} = \epsilon(-p\delta_{ij} + \nu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})). \tag{5.21}$$

Plugging Equations (5.14) - (5.21) into the expressions for the full hydrodynamic system and collecting terms of order ϵ yields the following linearized set of equations:

$$\frac{d\hat{u}}{dt} = -(1/\rho)\nabla\hat{P} + \nu\nabla^2\hat{u} \tag{5.22}$$

$$\nabla \cdot \hat{u} = 0 \tag{5.23}$$

$$\frac{d\hat{h}}{dt} = \hat{w} \tag{5.24}$$

$$-\tilde{P}/\rho + 2\nu\frac{\partial\hat{w}}{\partial z} = T\nabla^2\hat{h} \tag{5.25}$$

$$\frac{\partial\hat{v}}{\partial z} + \frac{\partial\hat{w}}{\partial x} = 0 \tag{5.26}$$

It is important to note the difference between $\hat{P}$ and $\tilde{P}$. $\hat{P}$ is the order ϵ component of the pressure in the body of the fluid, while $\tilde{P}$ is the order ϵ component of the pressure at the surface of the fluid. This difference is necessary, as the first expressions in the hydrodynamic system cited above hold in the body of the fluid, while the last three are surface conditions. This subtle difference in the pressures will not effect the unforced analysis presented in Chapter 6 since in the unforced case the pressure at the surface can be written:

$$\begin{aligned} P(x,0+z\epsilon,t) &\equiv \text{Order zero terms} + \epsilon\tilde{\mathrm{P}} \\ &= \bar{P}(x,0+z\epsilon,t) + \epsilon\hat{P}(x,0+z\epsilon,t) \\ &= P_o - \rho g 0 - \rho g z\epsilon + \epsilon\hat{P}(x,0+z\epsilon,t) \\ &\approx P_o + \epsilon\hat{P}(x,z,t) \end{aligned}$$

where gravity has been neglected and hence,

$$\tilde{P} = \hat{P}.$$

In the case where forcing is included however, this difference is important.

Chapter 6

Unforced Viscous Analysis

Before attacking the full forced problem, it is helpful to first consider the unforced case. This chapter focuses on solving the linearized partial differential equation system given in Chapter 5 for unforced solutions. These solutions are analyzed and interpreted in section 2 of this chapter.

6.1 Solving the Linearized System

Note that no x boundaries have been used in this problem. In other words, the fluid extends to infinite x in both directions, and hence is symmetric under translations of x. Hence, the x dependence of all of the physical functions can be written as e^{ikx} where k is real. Also, note that we are attempting to identify oscillatory solutions, and thus the time dependence of the physical parameters can be pulled out of each expression in the form of $e^{\lambda t}$ where λ is complex.

Consider the parameter vector,

$$\begin{bmatrix} \hat{v}(x,z,t) \\ \hat{w}(x,z,t) \\ \hat{P}(x,z,t) \\ \hat{h}(x,t) \end{bmatrix} = \begin{bmatrix} \hat{v}(z) \\ \hat{w}(z) \\ \hat{P}(z) \\ \hat{h}_o \end{bmatrix} e^{ikx+\lambda t},$$

where the x and t dependence of each quantity has been explicitly pulled out of the vector.

Plugging Equation (5.20) into Equation (5.21) yields,

$$\begin{aligned}\nabla \cdot \hat{u} &= 0 \\ &= ik\hat{v}(z)e^{ikx+\lambda t} + 0 + \frac{d\hat{w}}{dz}e^{ikx+\lambda t} \\ &= (ik\hat{v}(z) + \frac{d\hat{w}}{dz})e^{ikx+\lambda t},\end{aligned}$$

and hence,

$$-ik\hat{v} = \frac{d\hat{w}}{dz}$$

Let $\psi(z)$ be the stream function for this system. This function will act as the fluid's potential function, and provide a helpful way to relate the four functions in the parameter vector. Defining

$$-\psi ik = \hat{w},$$

then also yields

$$\frac{d\psi}{dz} = \hat{v}.$$

Using the linearized kinematic condition, Equation (5.24), allows h_o to be written in terms of ψ.

$$\begin{aligned}\frac{d\hat{h}}{dt} &= \frac{dh_o e^{ikx+\lambda t}}{dt} \\ &= \lambda h_o e^{ikx+\lambda t} \\ &= \hat{w}(x,0,t) \\ &= \hat{w}(0)e^{ikx+\lambda t},\end{aligned}$$

where $z = 0$ because this is a surface condition. Hence,

$$h_o = \hat{w}(0)/\lambda = \frac{-ik}{\lambda}\psi(0).$$

Observe that Equation (5.22) is a vector equation, and hence both x and z components must conform to the equation. The x component piece of Equation (5.21) yields a relationship between $\psi(z)$ and $\hat{P}$.

$$\begin{aligned}
\frac{d\hat{v}(x,z,t)}{dt} &= \lambda\hat{v}e^{ikx+\lambda t} \\
&= \lambda\frac{d\psi}{dz}e^{ikx+\lambda t} \\
&= -(1/\rho)\nabla\hat{P}(x,z,t)\cdot\hat{i}+\nu\nabla^2\tilde{u}(x,z,t) \\
&= (\frac{-ik}{\rho}\hat{P}(z)+\nu(\partial_{zz}-k^2)\hat{v}(z))e^{ikx+\lambda t} \\
&= (\frac{-ik}{\rho}\hat{P}(z)+\nu(\partial_{zz}-k^2)\frac{d\psi}{dz})e^{ikx+\lambda t}.
\end{aligned}$$

Hence,

$$\lambda\frac{d\psi}{dz}=\frac{-ik}{\rho}\hat{P}(z)+\nu(\partial_{zz}-k^2)\frac{d\psi}{dz}, \tag{6.1}$$

and further,

$$\hat{P}(z)=(\frac{-\rho\lambda}{ik}+\frac{\nu\rho}{ik}\partial_{zz}+\rho i\nu k)\frac{d\psi}{dz}.$$

Now the entire parameter vector can be written in terms of ψ.

$$\begin{bmatrix} \hat{v}(x,z,t) \\ \hat{w}(x,z,t) \\ \hat{P}(x,z,t) \\ \hat{h}(x,t) \end{bmatrix} = \begin{bmatrix} \hat{v}(z) \\ \hat{w}(z) \\ \hat{P}(z) \\ \hat{h}_o \end{bmatrix} e^{ikx+\lambda t} = \begin{bmatrix} \frac{d\psi}{dz} \\ -ik\psi \\ (\frac{-\rho\lambda}{ik}+\frac{\nu\rho}{ik}\partial_{zz}+\rho i\nu k)\frac{d\psi}{dz} \\ \frac{-ik}{\lambda}\psi(0) \end{bmatrix} e^{ikx+\lambda t}$$

Equation (5.21) can be used to solve for the functional form of $\psi(z)$ by first substituting ψ into the equation.

The x-component is given in equation (6.1). As for the z-component,

$$-ik\lambda\psi(z)=-(1/\rho)\frac{\partial\hat{P}(z)}{\partial z}-ik\nu(\partial_{zz}-k^2)\psi(z). \tag{6.2}$$

Taking a z derivative of Equation (6.1) and subtracting ik times Equation (6.2) removes the terms involving pressure from both equations, yielding the following ODE:

$$0 = (\partial_{zz} - k^2)(\nu(\partial_{zz} - k^2) - \lambda)\psi(z). \tag{6.3}$$

Note that this is a product of two second order ODEs, meaning that a function that makes either part of the product zero satisfies the equation. The general solution to this ODE is

$$\psi(z) = Ae^{kz} + Ce^{-kz} + Be^{mz} + De^{-mz},$$

where $m = \sqrt{\lambda/\nu + k^2}$. Since the velocity of the fluid approaches zero at the limit of infinite depth, $z \longrightarrow -\infty$, $C = D = 0$. This leaves

$$\psi(z) = Ae^{kz} + Be^{mz}. \tag{6.4}$$

Now with this function for $\psi(z)$, $\psi(z)$ can be substituted into the surface conditions to create a relation between λ and k. Using Equation (5.26), we see that:

$$\begin{aligned} 0 &= e^{ikx+\lambda t}\partial_{zz}\psi(0) + e^{ikx+\lambda t}k^2\psi(0) \\ &= 2k^2 A + (k^2 + m^2)B. \end{aligned}$$

Taking a similar approach to Equation (5.25) yields:

$$\begin{aligned} 0 &= (\frac{\lambda}{ik} - \frac{\nu}{ik}\partial_{zz} - ik\nu)\partial_z\psi(0) + 2ik\nu\partial_z\psi(0)\frac{-ik^3T}{\lambda}\psi(0) \\ &= (\frac{Tk^4}{\lambda} + k\lambda + 2\nu k^3)A + (\frac{k^4T}{\lambda} + m\lambda + 3\nu mk^2 - \nu m^3)B. \end{aligned}$$

These two surface conditions can be combined in a homogeneous matrix equation.

$$\begin{bmatrix} \frac{Tk^4}{\lambda} + k\lambda + 2\nu k^3 & \frac{k^4T}{\lambda} + m\lambda + 3\nu mk^2 - \nu m^3 \\ 2k^2 & k^2 + m^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In order for non-zero solutions for A and B to exist, the coefficient matrix in this homogeneous system must be singular, and hence must have determinant equal to zero. Therefore,

$$\begin{aligned} 0 &= (k^2+m^2)(\frac{Tk^4}{\lambda}+k\lambda+2\nu k^3)-2k^2(\frac{k^4T}{\lambda}+m\lambda+3\nu mk^2-\nu m^3) \\ &= 4k^2\lambda+4k^4\nu+\frac{Tk^3}{\nu}+\frac{\lambda^2}{\nu}-4k^3\nu\sqrt{\frac{\lambda}{\nu}+k^2} \end{aligned}$$

In order to remove the dimensions from this equation the constants ν and T are used to remove the dimension from k and λ. Let non-dimensional K and Λ be defined as follows:

$$\begin{aligned} \frac{T}{\nu^2}K &= k \\ \frac{T^2}{\nu^3}\Lambda &= \lambda. \end{aligned}$$

Multiplying the result of the determinant by $\frac{\nu^7}{T^4}$ gives:

$$0=4(\frac{\nu^4}{T^2}k^2)(\frac{\nu^3}{T^2}\lambda)+4\frac{\nu^8}{T^4}k^4+\frac{\nu^6}{T^3}k^3+\frac{\nu^6}{T^4}\lambda^2-4(\frac{\nu^6}{T^3}k^3)\sqrt{\frac{\nu^3}{T^2}\lambda+\frac{\nu^4}{T^2}k^2}$$

$$0=(\Lambda+2K^2)^2+K^3-4K^3\sqrt{\Lambda+K^2}.[11] \tag{6.5}$$

This polynomial can now be analyzed to determine the conditions under which growing solutions exist.

6.2 Analysis of Unforced Solutions

In this section, the physical ramifications of solutions to Equation (6.5) are investigated. Figure (6.1) presents the real parts of the roots of the polynomial. Note, only those solutions for which $-K^2 \leq Re\{\Lambda\}$, the real part of Λ, can be physical solutions. Recall that

$$\psi(z) = Ae^{kz} + Be^{mz},$$

from Equation (6.4), where $m = \sqrt{\lambda/\nu + k^2}$. If $\Lambda < -K^2$ then $\lambda/\nu + k^2 < 0$. This causes m to be complex, and $\psi(z)$ to carry an imaginary exponential, which indicates oscillatory fluid behavior in the z direction. In this unforced physical system, there is no oscillatory impulse given in the z direction, and the z direction does not admit to translational symmetry. Hence, oscillatory behavior will not occur in this direction, and solutions for which $\Lambda < -K^2$ are physically meaningless.

In figure (6.1), $\Lambda = -K^2$ is graphed with the thick line. Note that two of the solutions do not meet the physicality criteria while two of the solutions do. The imaginary parts of the solutions that have physical relevance are plotted in figure (6.2).

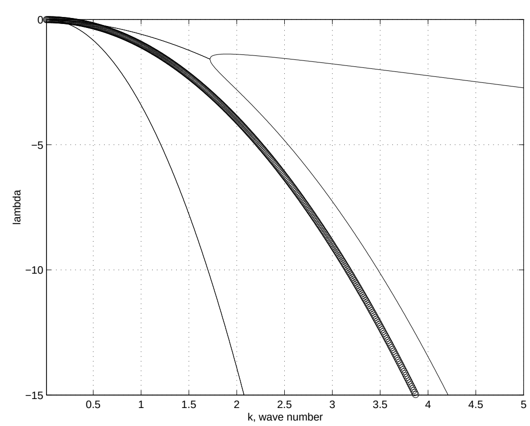

Figure 6.1: Plots of the real parts of the four solutions to Equation (6.5). The thick line is the physicality condition $\Lambda \geq -K^2$ which solutions must meet in order to be physically relevant. Note, the two physically meaningless solutions are a complex conjugate pair, and hence have the same real part.

Figures (6.1) and (6.2) show that the two physical solutions are a complex conjugate pair for small K. There is a bifurcation point at $K = 1.72$, where the solutions become strictly real valued, and these real values deviate from each other. For

$K > 1.72$ these are growing solutions, as they are strictly real valued. For $K < 1.72$ these solutions are oscillatory as caused by the complex exponent. Hence, growing solutions do not exist for $K < 1.72$.

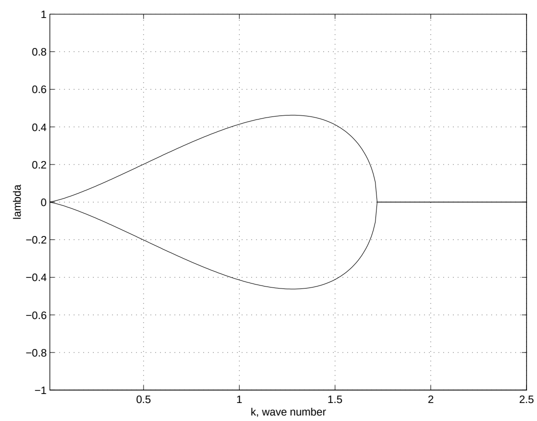

Figure 6.2: Plot of the imaginary parts of the physically meaningful solutions to Equation (6.5). Note that in order to be growing solutions, these must have no complex part. Hence, growing solutions only exist for $K > 1.72$.

For large K, scalings were determined for the growing solutions by applying a linear fit to a figure (6.3). Hence for $K > 1.72$,

$$K = -.9126\Lambda^2$$

and

$$K = -.5001\Lambda$$

are the scalings for the physically relevant solutions for the unforced problem.

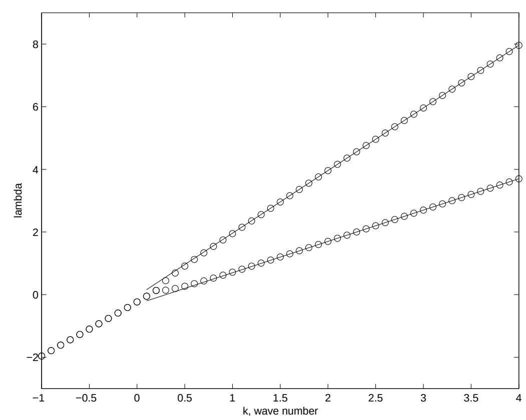

Figure 6.3: Log-log plot of physically meaningful solutions to Equation (6.5). Linear fit lines are presented, and correspond to the scalings given in the text.

Chapter 7

Excited Viscous Fluid Analysis

With the final blocks in place, now the full hydrodynamic system with forcing can be considered. The first three sections of this chapter apply many of the techniques highlighted previously in this thesis to the full problem, including non-dimensionalization, Floquet Theory, and Fourier methods, and methods for solving the unforced system from chapter 6. These sections closely follow Kumar's linear stability analysis. [9] The fourth section presents numerical solutions to the eigenvalue problem given by this system, and curves describing the boundary between growing and bounded solutions. The last section examines how the non-dimensional wave number K scales with both the non-dimensional forcing frequency Ω and the minimum required non-dimensional amplitude A.

7.1 Setting Up the Equations

Note that this is the order ϵ linear system, in 2 dimensions. Also note that the first two equations apply in the body of fluid, $z < 0$, while Equations (7.3) - (7.5) apply only at the surface $z = 0$.

$$\frac{d\hat{u}}{dt} = -(1/\rho)\nabla\hat{P} + \nu\nabla^2\hat{u} \tag{7.1}$$
$$\nabla \cdot \hat{u} = 0 \tag{7.2}$$
$$\frac{d\hat{h}}{dt} = \hat{w} \tag{7.3}$$
$$-\tilde{P}/\rho + 2\nu\frac{\partial\hat{w}}{\partial z} = (\sigma/\rho)\nabla^2\hat{h} \tag{7.4}$$
$$\frac{\partial\hat{v}}{\partial z} + \frac{\partial\hat{w}}{\partial x} = 0 \tag{7.5}$$

This is the same system given at the end of Chapter 5, with the exception that T has been replaced by σ/ρ. Note that this system is symmetric with respect to translations of the horizontal, or x, direction. Hence, the x-components of the fluid velocity $\hat{u}(x,z,t) =< \hat{v}(x,z,t), \hat{w}(x,z,t) >$, $\hat{h}(x,t)$ the fluid height, and the pressure of the fluid $\hat{P}(x,z,t)$, can be removed in the form of the exponential term e^{ikx}. In other words,

$$\begin{aligned}
\hat{u}(x,z,t) &= e^{ikx}\hat{u}(z,t) \\
&= e^{ikx} < \hat{v}(z,t), \hat{w}(z,t) > \\
\hat{h}(x,t) &= e^{ikx}\hat{h}(t) \\
\hat{P}(z,t) &= e^{ikx}\hat{P}(z,t)
\end{aligned}$$

Expanding Equation (7.2), yields:

$$
\begin{aligned}
\nabla \cdot \hat{u} &= 0 \\
&= \frac{\partial(e^{ikx}\hat{v}(z,t)}{\partial x} + \frac{\partial(e^{ikx}\hat{w}(z,t)}{\partial z} \\
&= ike^{ikx}\hat{v}(z,t) + e^{ikx}\frac{\partial \hat{w}(z,t)}{\partial z}.
\end{aligned}
$$

Thus,

$$
-ik\hat{v}(z,t) = \frac{\partial \hat{w}(z,t)}{\partial z} = \hat{w}_z(z,t). \tag{7.6}
$$

Writing the x components of Equation (7.1) separately, and applying Equation (7.6) to yields:

$$
\begin{aligned}
e^{ikx}\frac{d\hat{v}(z,t)}{dt} &= (i/k)e^{ikx}\frac{d\hat{w}_z(z,t)}{dt} \\
&= -(1/\rho)\frac{\partial(e^{ikx}\hat{P}(z,t))}{\partial x} + \nu\nabla^2(e^{ikx}\hat{v}(z,t)) \\
&= -(ik/\rho)e^{ikx}\hat{P}(z,t) + (i\nu/k)e^{ikx}(\partial_{zz} - k^2)\hat{w}_z(z,t),
\end{aligned}
$$

and thus

$$
(i/k)\frac{d\hat{w}_z(z,t)}{dt} = -(ik/\rho)\hat{P}(z,t) + (i\nu/k)(\partial_{zz} - k^2)\hat{w}_z(z,t). \tag{7.7}
$$

Writing the z component similarly gives:

$$
e^{ikx}\frac{\partial \hat{w}(z,t)}{dt} = -(1/\rho)e^{ikx}\frac{\partial(\hat{P}(z,t))}{\partial z} + \nu e^{ikx}(\partial_{zz} - k^2)\hat{w}(z,t),
$$

and hence

$$
\frac{\partial \hat{w}(z,t)}{dt} = -(1/\rho)\frac{\partial(\hat{P}(z,t))}{\partial z} + \nu(\partial_{zz} - k^2)\hat{w}(z,t). \tag{7.8}
$$

In order to remove $\hat{P}$ from these equations, consider combining Equations (7.7) and (7.8) by taking a z derivative of Equation (7.7) and subtracting ik times Equation (7.8) , as follows:

$$
\begin{aligned}
\partial_z((i/k)\frac{d\hat{w}_z(z,t)}{dt}) - (ik)\frac{\partial\hat{w}(z,t)}{dt} &= \partial_z(-(ik/\rho)\hat{P}(z,t) + (i\nu/k)(\partial_{zz} - k^2)\hat{w}_z(z,t)) \\
&+ (ik/\rho)\frac{\partial(\hat{P}(z,t))}{\partial z} - (ik\nu)(\partial_{zz} - k^2)\hat{w}(z,t) \\
&= \partial_z((i\nu/k)(\partial_{zz} - k^2)\hat{w}_z(z,t)) - (ik\nu)(\partial_{zz} - k^2)\hat{w}(z,t) \\
&= (\partial_{zz} - k^2)((i\nu/k)\partial_{zz}(\hat{w}(z,t)) - (ik\nu)\hat{w}(z,t)).
\end{aligned}
$$

Multiplying both sides by k/i gives:

$$
\begin{aligned}
(\partial_{zz} - k^2)(\frac{d\hat{w}(z,t)}{dt}) &= (\partial_{zz} - k^2)(\nu\partial_{zz}(\hat{w}(z,t)) - (k^2\nu)\hat{w}(z,t)) \\
&= (\partial_{zz} - k^2)((\partial_{zz} - k^2)\nu\hat{w}(z,t)),
\end{aligned}
$$

finally yielding

$$
(\partial_{zz} - k^2)(\partial_t - \nu(\partial_{zz} - k^2))\hat{w}(z,t) = 0. \tag{7.9}
$$

This equation will prove important in characterizing the solutions of $\hat{w}(z,t)$, though more information is needed about the time derivative of $\hat{w}(z,t)$ taking these solutions.

Another important condition on the solutions of $\hat{w}(z,t)$ can be found quickly by applying Equation (7.2) to Equation (7.5). Recall from Equation (7.2):

$$
\begin{aligned}
0 &= \nabla \cdot \hat{u} \\
&= \frac{\partial\hat{v}(x,z,t)}{\partial x} + \frac{\partial\hat{w}(x,z,t)}{\partial z},
\end{aligned}
$$

and hence

$$
-\frac{\partial\hat{v}(x,z,t)}{\partial x} = \frac{\partial\hat{w}(x,z,t)}{\partial z}. \tag{7.10}
$$

Taking an x derivative of Equation (7.5) gives:

$$\begin{aligned} 0 &= \frac{\partial \hat{v}_z(x,z,t)}{\partial x} + \frac{\partial \hat{w}_x(x,z,t)}{\partial x} \\ &= \frac{\partial \hat{v}_x(x,z,t)}{\partial z} + \frac{\partial \hat{w}_x(x,z,t)}{\partial x} \\ &= \frac{\partial \hat{w}_x(x,z,t)}{\partial x} - \frac{\partial \hat{w}_z(x,z,t)}{\partial z} \\ &= (-k^2)e^{ikx}\hat{w}(z,t) - e^{ikx}\partial_{zz}(\hat{w}(z,t)), \end{aligned}$$

and hence

$$(k^2 + \partial_{zz})\hat{w}(z,t) = 0. \tag{7.11}$$

It is important to note that this condition applies only at $z = 0$.

An important condition relating the vertical fluid velocity and the the fluid surface height can be fashioned by considering Equation (7.4). From the hydrostatic condition:

$$\vec{F} = (1/\rho)\nabla \bar{P},$$

where $\bar{P}$ is the order zero component of P and $\vec{F}$ is the external force per unit mass. In the case of periodic forcing,

$$\vec{F} = -(g - a\omega^2 \cos(\omega t))\hat{z} \approx -a\omega^2 \cos(\omega t)\hat{z},$$

where gravity has been ignored, and a is the maximum amplitude of the forcing, and not the maximum acceleration as used in Chapters 3 and 4. Substituting for $\vec{F}$, the hydrostatic condition yields:

$$\rho a\omega^2 \cos(\omega t)\hat{z} = \nabla \bar{P},$$

and further

$$\rho a\omega^2 \cos(\omega t) = \frac{\partial \bar{P}}{\partial z}.$$

Integrating gives,

$$\bar{P} = \rho a\omega^2 \cos(\omega t) z + P_o,$$

where P_o is the atmospheric pressure, $P_o = P(0)$.

Note that when linearizing about the hydrostatic condition, the fluid pressure can be written as

$$\begin{aligned} P(z,t) &= \bar{P}(\epsilon\hat{h}(t),t) + \epsilon\hat{P}(\epsilon\hat{h}(t),t) \\ &= P_o + \epsilon(\rho a\omega^2\cos(\omega t)\hat{h}(t) + \hat{P}), \end{aligned}$$

where $\hat{P}$ is order ϵ in the body of the fluid.

To order ϵ,

$$\tilde{P}(z,t) = \rho a\omega^2\cos(\omega t)\hat{h}(t) + \hat{P}(z,t).$$

Substituting this expression into Equation (7.4), yields:

$$\begin{aligned} -a\omega^2\cos(\omega t)\hat{h}(x,t) - \hat{P}(x,z,t)/\rho + 2\nu\frac{\partial\hat{w}(x,z,t)}{\partial z} &= (\sigma/\rho)\nabla^2\hat{h}(x,t) \\ &= (\sigma/\rho)\hat{h}_{xx}. \end{aligned}$$

Differentiating this equation with respect to x yields:

$$2\nu\frac{\partial\hat{w}_z(x,z,t)}{\partial x} - \tilde{P}_x(x,z,t)/\rho = (\sigma/\rho)\hat{h}_{xxx} + a\omega^2\cos(\omega t)\hat{h}_x(x,t).$$

Recall, that taking the x component of Equation (7.1) gives:

$$\hat{v}_t(x,z,t) = -\hat{P}_x/\rho + \nu\nabla^2\hat{v}(x,z,t).$$

Substituting in this expression for $\hat{P}_x$, eliminates pressure from the equation as follows:

$$2\nu\frac{\partial\hat{w}_z(x,z,t)}{\partial x} + \hat{v}_t(x,z,t) - \nu\nabla^2\hat{v}(x,z,t) = (\sigma/\rho)\hat{h}_{xxx} + a\omega^2\cos(\omega t)\hat{h}_x(x,t).$$

Differentiating with respect to x and then substituting $\hat{v}_x = -\hat{w}_z$ from Equation (7.2) removes the v dependence from the equation as shown:

$$2\nu \frac{\partial^2 \hat{w}_z(x,z,t)}{\partial x^2} - \frac{\partial \hat{w}_z(x,z,t)}{dt} + \nu \nabla^2 \hat{w}_z(x,z,t) = (\sigma/\rho)\hat{h}_{xxxx} + a\omega^2 \cos(\omega t)\hat{h}_{xx}(x,t).$$

Further simplifying this equation gives:

$$(\nu(3\partial_{xx} + \partial_{zz}) - \partial_t)\hat{w}_z(x,z,t) = (\sigma/\rho)\hat{h}_{xxxx}(x,t) + a\omega^2 \cos(\omega t)\hat{h}_{xx}(x,t).$$

Further, note that x derivatives of $w(x,z,t)$, and $h(x,t)$ only have the net effect of creating a factor of ik. Hence, this equation can be re-written as follows:

$$(\nu(-3k^2 + \partial_{zz}) - \partial_t)\hat{w}_z(z,t) = (\sigma/\rho)k^4\hat{h}(t) - ak^2\omega^2 \cos(\omega t)\hat{h}(t). \qquad (7.12)$$

7.2 Non-Dimensionalization

Equations (7.3), (7.9), (7.11), and (7.12) constitute the full hydrodynamic system, and allow the solutions to be characterized. Before proceeding on that road, it is prudent to non-dimensionalize the equations in this system. This is accomplished by first non-dimensionalizing time by setting $T = \omega t$, and using T as the new time variable. Note that this modulates all time derivatives by $\omega\frac{\partial}{\partial T} = \frac{\partial}{\partial t}$. Making this substitution, the full hydrodynamic system becomes:

$$(\partial_{zz} - k^2)(\omega\partial_T - \nu(\partial_{zz} - k^2))\hat{w}(z,T) = 0 \qquad (7.13)$$

$$(\nu(-3k^2 + \partial_{zz}) - \omega\partial_T)\hat{w}_z(z,T) = (\sigma/\rho)k^4\hat{h}(T) - ak^2\omega^2 \cos(T)\hat{h}(T) \qquad (7.14)$$

$$(k^2 + \partial_{zz})\hat{w}(z,T) = 0 \qquad (7.15)$$

$$\omega\frac{d\hat{h}}{dT} = \hat{w} \qquad (7.16)$$

In order to remove dimension from the equations it is necessary to define a length and time scales using some of the variables in the problem. For this task, ν and

σ/ρ are used and the appropriate scales are $l^* = \nu^2/(\sigma/\rho)$, and $t^* = \nu^3/(\sigma/\rho)^2$. Multiplying each equation by the appropriate dimension, ($t^{*2}l^*, t^{*2}, l^*t^*$, and t^*/l^* for equations (7.13)-(7.16) respectively) and applying the following non-dimensional variable definitions:

$$\begin{aligned} \frac{(\sigma/\rho)^2}{\nu^3}\Omega &= \omega \\ \frac{(\sigma/\rho)}{\nu^2}K &= k \\ \frac{\nu^2}{(\sigma/\rho)}A &= a, \end{aligned}$$

yields the following full hydrodynamic system:

$$(\nu/(\sigma/\rho))((\nu^4/(\sigma/\rho)^2)\partial_{zz} - K^2)(\Omega\partial_T - (\nu^4/(\sigma/\rho)^2)\partial_{zz} + K^2)\hat{w}(z,T) = 0 \quad (7.17)$$

$$(\nu^3/(\sigma/\rho)^2)(-3K^2+(\nu^4/(\sigma/\rho)^2)\partial_{zz}-\Omega\partial_T)\hat{w}_z(z,T) = ((\sigma/\rho)/\nu^2)(K^4+AK^2\Omega^2\cos(T))\hat{h}(T) \quad (7.18)$$

$$(\nu/(\sigma/\rho))(K^2 + (\nu^4/(\sigma/\rho)^2)\partial_{zz})\hat{w}(z,T) = 0 \quad (7.19)$$

$$\Omega((\sigma/\rho)/\nu^2)\frac{d\hat{h}}{dT} = (\nu/(\sigma/\rho))\hat{w}. \quad (7.20)$$

7.3 Floquet Solutions and the Eigenvalue Problem

The solutions for $\hat{w}(z,T)$ and $\hat{h}(T)$ can be written in Floquet form. Hence,

$$\hat{w}(z,T) = e^{\mu T}w_p(z,T) = e^{\mu T}\sum_{n=-\infty}^{\infty} w_n(z)e^{inT}$$

$$\hat{h}(T) = e^{\mu T}h_p(T) = e^{\mu T}\sum_{n=-\infty}^{\infty} h_n e^{inT},$$

where w_p, and h_p can be written as Fourier Series since they are 2π periodic. Note that since these series are 2π periodic, they describe harmonic solutions. In order to describe sub-harmonic solutions it is necessary to use the following series' of the form:

$$\hat{w}(z,T) = e^{\mu T} \sum_{n=-\infty,\text{odd}}^{\infty} w_n(z) e^{\frac{inT}{2}}$$

with odd n as explained in Chapter 4. The analysis that follows is not largely influenced by which series is used. Small distinctions do arise though, and these will be pointed out throughout the discussion. Also note, that in order for these series to be real, the coefficients must obey the following conditions:

$$w_1 = w_{-1}^*,$$

and

$$h_1 = h_{-1}^*$$

for the harmonic case. For the sub-harmonic case

$$w_0 = w_{-1}^*.$$

Further note that the $w_n(z)$ are only functions of z, and that the h_n are constants. Equation (7.17) can be used to solve for the form of the $w_n(z)$ functions, as the equation holds if:

$$0 = ((\nu^4/(\sigma/\rho)^2)\partial_{zz} - K^2)\hat{w}(z,T)$$

or

$$(\Omega\partial_T - (\nu^4/(\sigma/\rho)^2)\partial_{zz} - K^2)\hat{w}(z,T) = 0.$$

Substituting the Fourier expanded form of $\hat{w}(z,T)$ into these equations give the following sets of solutions

$$w_n(z) = a_n e^{((\sigma/\rho)/\nu^2)Kz} + b_n e^{-((\sigma/\rho)/\nu^2)Kz},$$

and

$$w_n(z) = c_n e^{((\sigma/\rho)/\nu^2)Q_n z} + d_n e^{-((\sigma/\rho)/\nu^2)Q_n z},$$

where Q_n is the positive root of $Q_n^2 = K^2+\Omega(\mu+in)$. Note, that Q_n is slightly different in the sub-harmonic case, where n is replaced by $n/2$ for odd n only. Applying the condition that as $z \to -\infty$, $w_n(z) \to 0$, forces $b_n = d_n = 0$. Hence,

$$w_n(z) = a_n e^{((\sigma/\rho)/\nu^2)Kz} + c_n e^{((\sigma/\rho)/\nu^2)Q_n z} \tag{7.21}$$

Substituting the Floquet/Fourier definitions of $\hat{w}$ and $\hat{h}$ into Equation (7.18) significantly simplifies the expression:

$$\frac{\nu^3}{(\sigma/\rho)^2}(-3K^2 + \frac{\nu^4}{(\sigma/\rho)^2}\partial_{zz} - \Omega\partial_T)\hat{w}_z(z,T) = \frac{(\sigma/\rho)}{\nu^2}(K^4 + AK^2\Omega^2\cos(T))\hat{h}(T)$$

$\frac{\nu^5}{(\sigma/\rho)^3}e^{\mu T}\sum_{n=-\infty}^{\infty}(\frac{\nu^4}{(\sigma/\rho)^2}\partial_{zzz} - (3K^2 + \Omega(in+\mu))\partial_z)w_n(z)e^{inT} =$

$e^{\mu T}\sum_{n=-\infty}^{\infty} h_n(K^4 e^{inT} + \frac{AK^2\Omega^2}{2}e^{i(n+1)T} + \frac{AK^2\Omega^2}{2}e^{i(n-1)T})$

$\frac{\nu^5}{(\sigma/\rho)^3}e^{\mu T}\sum_{n=-\infty}^{\infty}(\frac{\nu^4}{(\sigma/\rho)^2}\partial_{zzz} - (2K^2 + Q_n^2)\partial_z)w_n(z)e^{inT} =$

$e^{\mu T}\sum_{n=-\infty}^{\infty}(K^4 h_n + \frac{AK^2\Omega^2}{2}h_{n-1} + \frac{AK^2\Omega^2}{2}h_{n+1})e^{inT}.$

For each n,

$$\frac{\nu^5}{(\sigma/\rho)^3}(\frac{\nu^4}{(\sigma/\rho)^2}\partial_{zzz} - (2K^2 + Q_n^2)\partial_z)w_n(z) = (K^4 h_n + \frac{AK^2\Omega^2}{2}h_{n-1} + \frac{AK^2\Omega^2}{2}h_{n+1}) \tag{7.22}$$

Equations (7.19) and (7.20) can be used to write $\partial_z w_n(0)$, and $\partial_{zzz} w_n(0)$ as multiples of h_n. First to get a relation between a_n and c_n, Equation (7.19) will be used.

$$\begin{aligned} 0 &= (\nu/(\sigma/\rho))(K^2 + (\nu^4/(\sigma/\rho)^2)\partial_{zz})e^{\mu T}\sum_{n=-\infty}^{\infty} w_n(z)e^{inT} \\ 0 &= \sum_{n=-\infty}^{\infty}(K^2 + (\nu^4/(\sigma/\rho)^2)\partial_{zz})w_n(z)e^{inT} \\ &= \sum_{n=-\infty}^{\infty}(K^2 + (\nu^4/(\sigma/\rho)^2)\partial_{zz})(a_n e^{((\sigma/\rho)/\nu^2)Kz} + c_n e^{((\sigma/\rho)/\nu^2)Q_n z})e^{inT} \\ &= \sum_{n=-\infty}^{\infty}(2a_n K^2 e^{((\sigma/\rho)/\nu^2)Kz} + (K^2+Q_n^2)c_n e^{((\sigma/\rho)/\nu^2)Q_n z})e^{inT} \end{aligned}$$

Equation (7.19) holds at the surface, $z = 0$. Applying this restriction to the equation yields:

$$0 = \sum_{n=-\infty}^{\infty}(2a_n K^2 + (K^2+Q_n^2)c_n)e^{inT}.$$

Further, since each e^{inT} is independent,

$$0 = 2a_n K^2 + (K^2+Q_n^2)c_n. \tag{7.23}$$

Note that this is true for all n. Equation (7.20) is also a surface condition and relates a_n, and c_n to h_n as follows:

$$\begin{aligned} \Omega((\sigma/\rho)/\nu^2)\frac{d\hat{h}}{dT} &= (\nu/(\sigma/\rho))\hat{w} \\ \Omega((\sigma/\rho)/\nu^2)e^{\mu T}\sum_{n=-\infty}^{\infty}(\mu+in)h_n e^{inT} &= (\nu/(\sigma/\rho))e^{\mu T}\sum_{n=-\infty}^{\infty}(a_n+c_n)e^{inT}. \end{aligned}$$

Hence, for each n,

$$\Omega((\sigma/\rho)^2/\nu^3)(\mu+in)h_n e^{inT} = (a_n+c_n)e^{inT}$$

$$(\Omega(\sigma/\rho)^2(\mu+in)/\nu^3)h_n = a_n + c_n. \tag{7.24}$$

Combining Equations (7.23) and (7.24) yields the following relations between h_n and

a_n, and h_n and c_n.

$$\begin{aligned}\frac{\Omega(\sigma/\rho)^2(\mu+in)}{\nu^3(1-\frac{2K^2}{K^2+Q_n^2})}h_n &= a_n\\ \frac{\Omega(\sigma/\rho)^2(\mu+in)}{\nu^3(1-\frac{K^2+Q_n^2}{2K^2})}h_n &= c_n.\end{aligned}$$

Plugging these equations in for a_n and c_n, yields formulas for $\partial_z w_n(0)$, and $\partial_{zzz} w_n(0)$ in terms of h_n.

$$\begin{aligned}\partial_z w_n(0) &= a_n((\sigma/\rho)/\nu^2)K + c_n((\sigma/\rho)/\nu^2)Q_n\\ &= (\frac{K\Omega(\sigma/\rho)^3(\mu+in)}{\nu^5(1-\frac{2K^2}{K^2+Q_n^2})} + \frac{Q_n\Omega(\sigma/\rho)^3(\mu+in)}{\nu^5(1-\frac{K^2+Q_n^2}{2K^2})})h_n\\ &= (\frac{(\sigma/\rho)^3}{\nu^5}(K^3+KQ_n^2-2Q_nK^2))h_n\end{aligned}$$

and

$$\begin{aligned}\partial_{zzz} w_n(0) &= (\frac{K^3\Omega(\sigma/\rho)^5(\mu+in)}{\nu^9(1-\frac{2K^2}{K^2+Q_n^2})} + \frac{Q_n^3\Omega(\sigma/\rho)^5(\mu+in)}{\nu^9(1-\frac{K^2+Q_n^2}{2K^2})})h_n\\ &= (\frac{(\sigma/\rho)^5}{\nu^9}(K^5+K^3Q_n^2-2Q_n^3K^2))h_n\end{aligned}$$

Plugging these formulas into Equation (7.22) gives a relation between h_n, h_{n-1}, and h_{n+1}.

$$(K^5+K^3Q_n^2-2Q_n^3K^2-(2K^2+Q_n^2)(K^3+KQ_n^2-2Q_nK^2)-K^4)h_n = \frac{-A\Omega^2K^2}{2}(h_{n+1}+h_{n-1}) \tag{7.25}$$

Let $K^5+K^3Q_n^2-2Q_n^3K^2-(2K^2+Q_n^2)(K^3+KQ_n^2-2Q_nK^2)-K^4 = B_n^r+B_n^i = C_n$

Observe that the reality conditions on the Fourier series' give a convenient way to restrict the series' to non-negative n for both harmonic and sub-harmonic waves. Using these reality conditions, the following matrix equations can be setup for the harmonic and sub-harmonic cases.

$$(7.26)\begin{bmatrix} B_0^r & -B_0^i & 0 & 0 & 0 & 0 & \\ B_0^i & B_0^r & 0 & 0 & 0 & 0 & \\ 0 & 0 & B_1^r & -B_1^i & 0 & 0 & \\ 0 & 0 & B_1^i & B_1^r & 0 & 0 & \\ 0 & 0 & 0 & 0 & B_2^r & -B_2^i & \\ 0 & 0 & 0 & 0 & B_2^i & B_2^r & \\ & & & & & & \ddots \end{bmatrix} \begin{bmatrix} h_0^r \\ h_0^i \\ h_1^r \\ h_1^i \\ h_2^r \\ h_2^i \\ \vdots \end{bmatrix} =$$

$$\frac{-A\Omega^2 K^2}{2} \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & \\ & & & & & & \ddots \end{bmatrix} \begin{bmatrix} h_0^r \\ h_0^i \\ h_1^r \\ h_1^i \\ h_2^r \\ h_2^i \\ \vdots \end{bmatrix}$$

for the harmonic case, and

$$(7.27)\begin{bmatrix} B_0^r & -B_0^i & 0 & 0 & 0 & 0 & \\ B_0^i & B_0^r & 0 & 0 & 0 & 0 & \\ 0 & 0 & B_1^r & -B_1^i & 0 & 0 & \\ 0 & 0 & B_1^i & B_1^r & 0 & 0 & \\ 0 & 0 & 0 & 0 & B_2^r & -B_2^i & \\ 0 & 0 & 0 & 0 & B_2^i & B_2^r & \\ & & & & & & \ddots \end{bmatrix} \begin{bmatrix} h_0^r \\ h_0^i \\ h_1^r \\ h_1^i \\ h_2^r \\ h_2^i \\ \vdots \end{bmatrix} =$$

$$\frac{-A\Omega^2K^2}{2}\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & \\ 0 & -1 & 0 & 1 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & \\ & & & & & & \ddots \end{bmatrix}\begin{bmatrix} h_0^r \\ h_0^i \\ h_1^r \\ h_1^i \\ h_2^r \\ h_2^i \\ \vdots \end{bmatrix}$$

for the sub-harmonic case.

7.4 Stability Analysis

The matrix equation can be solved as an eigenvalue problem of the form $Ax = \lambda Bx$. Setting the Floquet exponent, μ, to zero explicitly determines the matrix A for both cases. It is appropriate to set $\mu = 0$ as this is the case when the solution is periodic, which was an assumption necessary to approximate solutions with Fourier series to this analysis. Solving for the eigenvalues of the matrix equation, and plotting these values yields the following plot for $\Omega = 100000$.

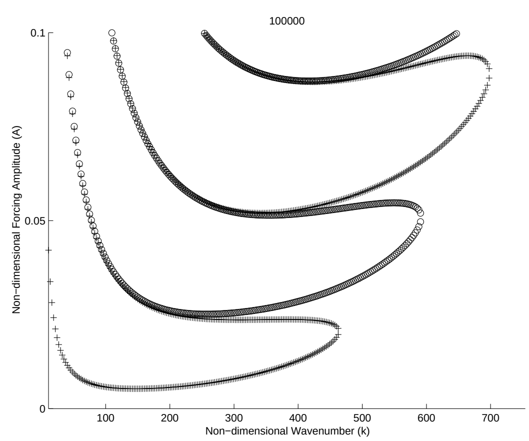

Figure 7.1: Plot of K vs. A for $\Omega = 100000$, where the plotted points correspond to eigenvalues of Equations (7.26) and (7.27). Note that the + symbols correspond to sub-harmonic solutions, while the o symbols correspond to harmonic solutions.

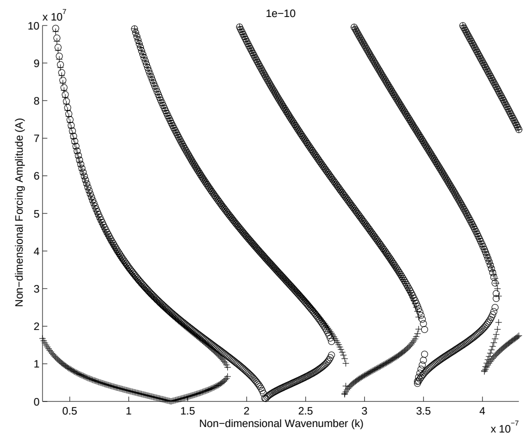

Figure 7.2: Plot of K vs. A for $\Omega = 10^{-10}$, where the plotted points correspond to eigenvalues of Equations (7.26) and (7.27). Note that the + symbols correspond to sub-harmonic solutions, while the o symbols correspond to harmonic solutions. In the low frequency limit, these stability curves approach the stability curves for the inviscid Mathieu analysis in Chapter 3.

7.5 K Scalings

The main goal of this work was to compute the relationship between the surface wave number and the forcing frequency. With the stability curves provided in section 7.4, this is now possible. Stability curves were produced for values of Ω between 10^-15 and 10^10, and on each plot the minimum amplitude needed to cause instability was found along with the wave number value where that minimum occurred. Log-log plots of wave number vs. minimum amplitude and forcing frequency vs. wave number are given in figures (7.3) and (7.4).

In figure (7.3), note that 2 distinct regimes exist. Using a linear curve fit, power laws and regime ranges can be discerned. For large K, $K > .378$,

$$a = .7707k^{-1},$$

corresponding to the blue line. For small values of K, $.378 > K$,

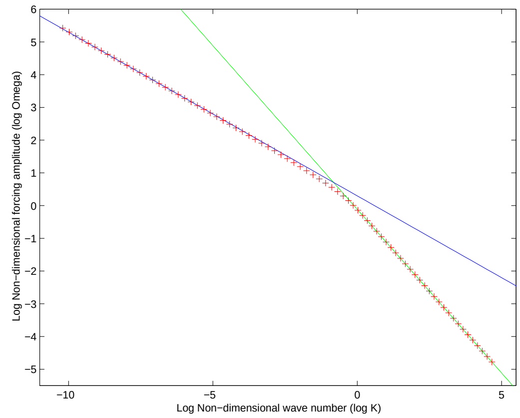

Figure 7.3: Log-log plot of non dimensional minimum forcing required for instability versus non dimensional wave number. Linear fits for the data are graphed as given in the text.

$$a = 1.253 \frac{\nu}{(\sigma/\rho)^{\frac{1}{2}}} k^{\frac{-1}{2}},$$

which is given by the green line. Figure (7.4) similarly exhibits two separate K and Ω scalings regimes. For large K, $K > .2068$,

$$k = .47206 \nu^{\frac{-1}{2}} \omega^{\frac{1}{2}},$$

as shown by the blue line while for $K < .2068$,

$$k = .62616 (\sigma/\rho)^{\frac{-1}{3}} \omega^{\frac{2}{3}},$$

corresponding to the green line. Note that the values of K used to define these ranges were taken from the intersection point of the linear fits performed on figures (7.3) and (7.4). Hence, these are approximate ranges, and the $K = .2068$ transition in figure (7.4) corresponds to the $K = .378$ transition in figure (7.3).

To better understand these regimes, it is important to observe which constant factors matter in which regime. For large Ω, viscosity determines the wave number, while

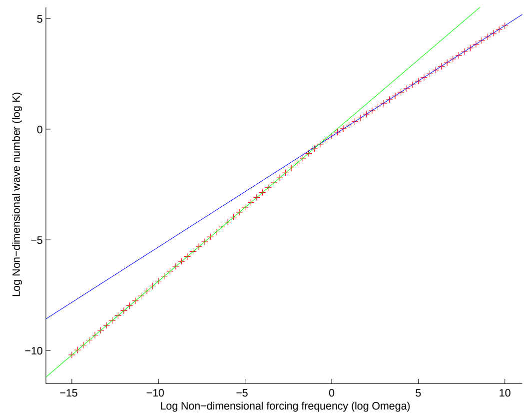

Figure 7.4: Log-log plot of non dimensional wave number vs. non dimensional forcing frequency. Linear fits for the data are graphed as given in the text.

for small Ω viscosity is dominated by surface tension in determining the wave number.

This understanding of these scalings and precise statement of the results are important and will hopefully be of immediate use in precisely identifying the constant relating surface wavelength to droplet size.

Chapter 8

Conclusion

In this work, Faraday Excitation of highly viscous fluids was studied with linear stability analysis of the full hydrodynamic system. For large K, where K is the non-dimensional wave number, it was found that:

$$k = .47206\nu^{\frac{-1}{2}}\omega^{\frac{1}{2}},$$

where ω is the forcing frequency, and ν is the fluid viscosity. For small K,

$$k = .62616(\sigma/\rho)^{\frac{-1}{3}}\omega^{\frac{2}{3}},$$

where σ is the surface tension and ρ is the fluid density.

These scalings are being used to determine the constant relating the size of particles ejected from a surface wave under Faraday Excitation to the wavelength of that wave. This is described in more detail in section (1.1). Figure (8.1) presents preliminary results of the ultrasonic atomization as compared to the theoretical predictions for the wavelength of surface waves as a function of non dimensional forcing frequency. The four data points predict a value for the constant multiplier to the surface wave wavelength that yields particle radius of .154. This value has large error, though, since there are currently only 4 data points. Note that all of the current data points test the low Ω regime. In future work a more detailed analysis of this experiment will be given. Additionally, data points to test a wider range of the Ω values will be presented. Hopefully this will include values in the large Ω range.

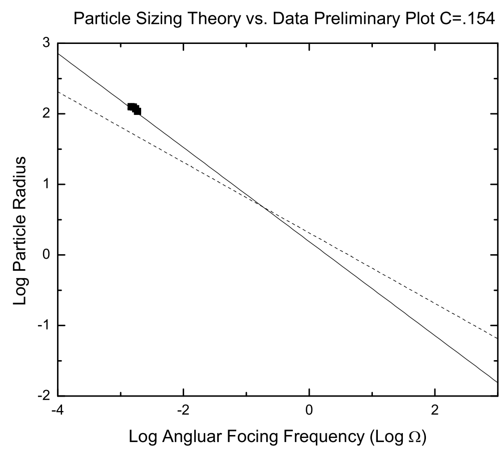

Figure 8.1: Log-log plot comparing particle sizing data points to theoretical predictions. The data points shown are of the non-dimensional particle radius (y axis) vs. the given non-dimensional oscillating frequency (x axis). The lines are the log scale predictions for the non-dimensional surface wavelength of excited waves as a function of non-dimensional forcing frequency. Note that the solid line is the low frequency scaling, while the dashed line is the high frequency scaling. The data gives a value for the constant multiplying the surface wavelength to yield particle radius of $c = .154$, and hence the scalings were appropriately shifted for this plot. This is only very perliminary data, and consequently the error in the value of the constant is large since only 4 data points have been taken. However, the error on each data point is small, error bars are shown in this plot.

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