

Systems, Flows and Stability

We are going to consider

$$\dot{x} = f(x), \tag{1}$$

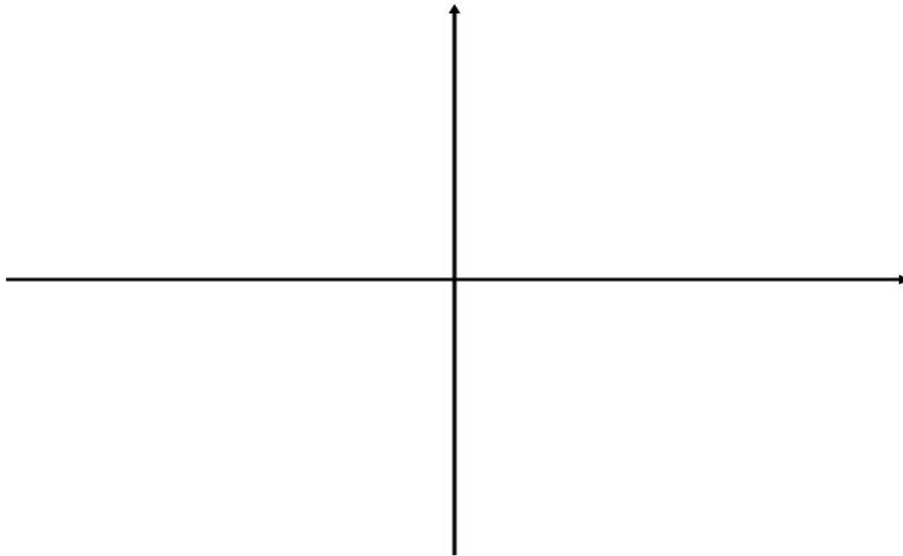
where x is a continuous function of t and $f(x)$ is a *smooth* function of x and does not depend explicitly on t . In Strogatz terminology, we can refer to equation 1 as a **first-order system**—this is in the sense of a dynamical system and not a system of equations.

More specifically, let's consider $\dot{x} = \sin(x)$. This differential equation has the implicit solution:

$$t = \ln \left| \frac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)} \right|.$$

How helpful is this solution in terms of explaining what happens to x as t changes?

We use a different approach. Let's start by sketching \dot{x} as a function of x on the following graph:



For physical intuition, let's consider x as the position of a particle on the real line. Then,

What is \dot{x} ?

When does the flow move to the right? to the left?

What happens at points where $x = n\pi$?

About those fixed points:

Which ones are stable?

Which ones are unstable?

What might be a half-stable or hybrid case?

Using the information above, sketch some trajectories of x versus t on the following graph:



For You!

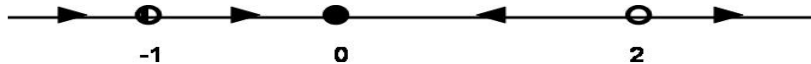
*Exercise 1.*¹ Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions.

(a) $\dot{x} = x - x^3$

(b) $\dot{x} = x^2(6 - x)$

(c) $\dot{x} = 0$

*Exercise 2.*¹ (Working backwards, from flows to equations) Given an equation $f(x)$, we know how to sketch the corresponding flow on the real line. Here you are asked to solve the opposite problem: for the following phase portrait, find an equation that is consistent with it. (There are an infinite number of correct answers—and wrong ones too!)



*Exercise 3.*¹ The growth of cancerous tumors can be modeled by the Gompertz law $\dot{N} = -aN \ln(bN)$, where $N(t)$ is proportional to the number of cells in the tumor and $a, b > 0$ are parameters.

(a) Interpret a and b biologically.

(b) Sketch the vector field and then graph $N(t)$ for various initial values. The predictions of this simple model agree surprisingly well with data on tumor growth, as long as N is not too small; see Aroesty et al. (1973) and Newton (1980) for examples.

*Exercise 4.*² Another simple model of tumor/host interaction describes the growth of two populations, each growing according to a logistic law and competing with each other for resources. In this model, we lump together all non-tumor cells which are at

¹Strogatz, 1994

²de Pillis and Radunskaya, 2004

the tumor site, including normal tissue as well as immune cells. We do *not* assume a constant source of immune cells.

Let $X(t)$ denote the normal cell population at time t , including immune cells, and let $Y(t)$ denote the tumor population at time t . The system of differential equations which describes the model is:

$$\begin{aligned}\frac{dX}{dt} &= a_1X(1 - b_1X) - c_1XY \\ \frac{dY}{dt} &= a_2Y(1 - b_2Y) - c_2XY\end{aligned}$$

(a) What is the biological interpretation of each of the parameters a_1, a_2, b_1, b_2, c_1 and c_2 ? Are they necessarily all positive or negative?

(b) Describe hypothetical experiments which would allow the determination of these parameters.

References

1. de Pillis, L.G. and Radunskaya, A.E. *Exercises for Tumor Dynamics Module*. 2004.
2. Strogatz, S. *Nonlinear Dynamics and Chaos*. Perseus Books Publishing, LLC., 1994.