DAY 3

Linearization! SOLUTIONS

1 (a) $\dot{x} = x - y$; $\dot{y} = x^2 - 4$

<u>fixed points:</u> $\dot{x} = 0 = x - y \Rightarrow x = y$

$\dot{y} = 0 = x^2 - 4$

$= (x+2)(x-2)$

So $\dot{y} = 0$ only when $x = \pm 2$ which means the equilibrium points are $(2,2)$, $(-2,-2)$

<u>Jacobian:</u> $f(x,y) = x - y$

$g(x,y) = x^2 - 4$

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix}$$

$J(2,2) = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$ w/ evalues $\lambda_1, \lambda_2 = \dfrac{1 \pm i\sqrt{15}}{2}$

★ $(2,2)$ is an unstable spiral.

$J(-2,-2) = \begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix}$ w/ evalues $\lambda_1, \lambda_2 = \dfrac{1 \pm \sqrt{17}}{2}$

★ $(-2,-2)$ is a saddle point

(b) $\dot{x} = \sin y$; $\dot{y} = x - x^3$

<u>fixed points:</u> $\dot{x} = 0 = \sin y \Rightarrow y = \pm n\pi$

$\dot{y} = 0 = x(1 - x^2) \Rightarrow x = 0, \pm 1$

the equilibrium points are $(0, \pm n\pi)$, $(1, \pm n\pi)$, $(-1, \pm n\pi)$

<u>Jacobian:</u> $J = \begin{bmatrix} 0 & \cos y \\ 1 - 3x^2 & 0 \end{bmatrix}$

$J(0, n\pi) = \begin{bmatrix} 0 & \cos n\pi \\ 1 & 0 \end{bmatrix}$ → for n odd $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ → for n even $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

for n odd the evalues are ± 1, so these points are saddle points; for n even the evalues are $\pm i$ so the linearization says these points are centers

$$J(\pm 1, n\pi) = \begin{bmatrix} 0 & \cos n\pi \\ -2 & 0 \end{bmatrix} \begin{array}{l} \nearrow \text{for } n \text{ odd: } \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \\ \searrow \text{for } n \text{ even: } \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \end{array}$$

for n odd the e-values are $\pm i\sqrt{2}$, so the linearization predicts these points are centers; for n even the e-values are $\pm\sqrt{2}$ so these are saddle points.

(c) $\dot{x} = 1 + y - e^{-x}$; $\dot{y} = x^3 - y$

<u>fixed points</u>: $\dot{x} = 0 = 1 + y - e^{-x}$ $\quad y = e^{-x} - 1$

$\dot{y} = 0 = x^3 - y$ $\quad x^3 = y$

the equilibrium point is $(0,0)$

<u>Jacobian</u>: $J = \begin{bmatrix} e^{-x} & 1 \\ 3x^2 & -1 \end{bmatrix}$

$J(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ w/ evalues ± 1

★ $(0,0)$ is a saddle point.

(d) $\dot{x} = y + x - x^3$, $\dot{y} = -y$

<u>fixed points</u>: $\dot{x} = 0 = y + x - x^3 \to 0 + x - x^3 = 0 = x(1-x^2)$

$\dot{y} = 0 = -y \Rightarrow y = 0$

the equilibrium points are $(0,0)$, $(\pm 1, 0)$

<u>Jacobian</u>: $J = \begin{bmatrix} 1-3x^2 & 1 \\ 0 & -1 \end{bmatrix}$

$J(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ w/ evalues ± 1

★ $(0,0)$ is a saddle point.

$J(\pm 1, 0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$ w/ evalues $-2, -1$

★ $(\pm 1, 0)$ are both stable nodes.

(e) $\dot{x} = \sin y$, $\dot{y} = \cos x$

<u>fixed points</u>: $\dot{x} = 0 = \sin y \Rightarrow y = k\pi$

$\dot{y} = 0 = \cos x \Rightarrow x = \frac{2n+1}{2}\pi$

The equilibrium points are $(n/2\pi, k\pi)$

<u>Jacobian</u>: $J = \begin{bmatrix} 0 & \cos y \\ -\sin x & 0 \end{bmatrix}$

$J\left(\frac{2n+1}{2}\pi, k\pi\right) = \begin{bmatrix} 0 & \cos(k\pi) \\ -\sin\left(\frac{2n+1}{2}\pi\right) & 0 \end{bmatrix}$

4 cases:

(i) k odd and $\frac{2n+1}{2} = 1, 5, 9 \ldots \Rightarrow J = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ w/ evalues ± 1

these points are saddles.

(ii) k odd and $\frac{2n+1}{2} = 3, 7, 11 \ldots \Rightarrow J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ w/ evalues $\pm i$

these points are predicted to be centers.

(iii) K even and $\frac{2n+1}{2} = 1, 5, 9 \ldots \Rightarrow J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ w/ evalues $\pm i$

these points are also predicted to be centers.

(iv) k even and $\frac{2n+1}{2} = 3, 7, 11, \ldots \Rightarrow J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ w/ evalues ± 1

these points are saddles.

(f) $\dot{x} = xy - 1$; $\dot{y} = x - y^3$

<u>fixed points</u>: $\dot{x} = 0 \Rightarrow xy = 1 \rightarrow y^4 = 1$ $y = \pm 1, \pm i$

$\dot{y} = 0 \Rightarrow x = y^3$

the equilibrium points are $(1,1)$, $(-1,-1)$, $(-i, i)$, $(i, -i)$

since we're considering points in $\mathbb{R}^2$, we consider $(1,1)$ $(-1,-1)$

<u>Jacobian</u>: $J = \begin{bmatrix} y & x \\ 1 & -3y^2 \end{bmatrix}$

$J(1,1) = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$ w/ e-values $-1 \pm \sqrt{5}$

★ $(1,1)$ is a saddle point.

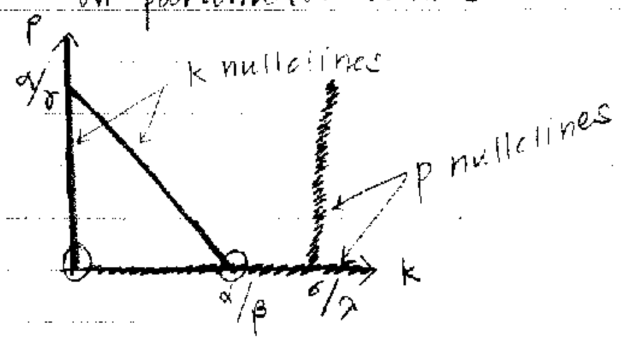
$J(-1,-1) = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$ w/ e-values $-2, -2$

★ $(-1,-1)$ is a stable node.

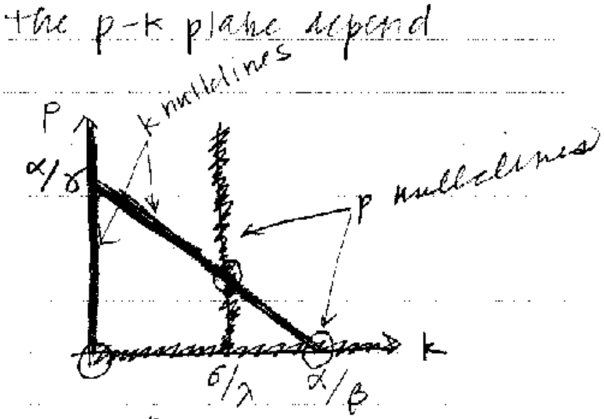
2. $\dot{k} = \alpha k - \beta k^2 - \gamma k p$
$\dot{p} = -\sigma p + \lambda k p$

(i) nullclines: $\dot{p} = 0 \Rightarrow p(-\sigma + \lambda k) = 0$
$p = 0$ or $k = \sigma/\lambda$
$\dot{k} = 0 \Rightarrow \alpha k - \beta k^2 - \gamma k p = 0$
$k(\alpha - \beta k - \gamma p) = 0$
either $k = 0$ or $\alpha - \beta k - \gamma p = 0$
$p = (\alpha - \beta k)/\gamma$

Note that the nullclines positions in the p-k plane depend on parameter values.



here $\alpha/\beta < \sigma/\lambda$



here $\alpha/\beta > \sigma/\lambda$

(ii) equilibrium points (at intersection of nullclines)
when $\alpha/\beta < \sigma/\lambda$. $(0,0)$ $(\alpha/\beta, 0)$
when $\alpha/\beta > \sigma/\lambda$. $(0,0)$ $(\alpha/\beta, 0)$ $(\sigma/\lambda, \alpha/\gamma - \beta\sigma/\gamma\lambda)$

* assume pos. parameters

(iii) Onto the Stability by looking at the Jacobian.

$$J = \begin{bmatrix} \partial f/\partial k & \partial f/\partial p \\ \partial g/\partial k & \partial g/\partial p \end{bmatrix} = \begin{bmatrix} \alpha - 2\beta k - \gamma p & -\gamma k \\ \lambda p & -\sigma + \lambda k \end{bmatrix}$$

* case (i) $\alpha/\beta < \sigma/\lambda$:

$J(0,0) = \begin{bmatrix} \alpha & 0 \\ 0 & -\sigma \end{bmatrix}$ e values are $\alpha, -\sigma$

(0,0) is a saddle

$J(\alpha/\beta, 0) = \begin{bmatrix} -\alpha & -\alpha\gamma/\beta \\ 0 & -\sigma + \lambda\alpha/\beta \end{bmatrix}$ evalues are $-\alpha$, $-\sigma + \lambda\alpha/\beta$

* the second evalue is negative when

$$-\sigma + \lambda\alpha/\beta < 0$$
$$-\beta\sigma + \lambda\alpha < 0$$
$$\lambda\alpha < \beta\sigma$$
$$\alpha/\beta < \sigma/\lambda$$ * and this is true in this case!

$(\alpha/\beta, 0)$ is a stable node

* case (ii) $\alpha/\beta > \sigma/\lambda$

$J(0,0) = \begin{bmatrix} \alpha & 0 \\ 0 & -\sigma \end{bmatrix}$ e values are $\alpha, -\sigma$

(0,0) is still a saddle!

$J(\alpha/\beta, 0) = \begin{bmatrix} -\alpha & -\alpha\gamma/\beta \\ 0 & -\sigma + \lambda\alpha/\beta \end{bmatrix}$ evalues are $-\alpha$, $-\sigma + \lambda\alpha/\beta$

in this case we know $\alpha/\beta > \sigma/\lambda$ (see above!)

So the 2nd evalue is now positive

and $(\alpha/\beta, 0)$ is a saddle point.

onto the 3rd eq. point. (w/o e-values!)

$$J(\sigma/\lambda,\ \alpha/\gamma - \beta\sigma/\gamma\lambda)$$

$$= \begin{bmatrix} \alpha - 2\beta(\sigma/\lambda) - \gamma(\alpha/\gamma - \beta\sigma/\gamma\lambda) & -\gamma(\sigma/\lambda) \\ \lambda(\alpha/\gamma - \beta\sigma/\gamma\lambda) & -\sigma + \lambda(\sigma/\lambda) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha - 2\beta\sigma/\lambda - \alpha + \beta\sigma/\lambda & -\gamma\sigma/\lambda \\ \lambda\alpha/\gamma - \beta\sigma/\gamma & -\sigma + \sigma \end{bmatrix}$$

$$= \begin{bmatrix} -\beta\sigma/\lambda & -\sigma\gamma/\lambda \\ (\lambda\alpha - \beta\sigma)/\gamma & 0 \end{bmatrix}$$

$\text{trace } J = \frac{-\beta\sigma}{\lambda}$ $\quad \det J = \frac{\alpha\cancel{\gamma}}{\lambda}\left(\frac{\lambda\alpha - \beta\sigma}{\cancel{\gamma}}\right)$

$$= \frac{\alpha}{\lambda}(\lambda\alpha - \beta\sigma)$$

$\det J$ is positive if $\lambda\alpha - \beta\sigma > 0$

$$\lambda\alpha > \beta\sigma$$

$$\alpha/\beta > \sigma/\lambda$$

this is our scenario!

So we know this third fixed point is stable.

3. $\dot{x} = -y - x^2$; $\dot{y} = x$

We want to know if $(0,0)$ is a nonlinear center.

First check that it is a linear center:

$J = \begin{bmatrix} -2x & -1 \\ 1 & 0 \end{bmatrix}$ $\quad J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ w/ evalues $\pm i$

So the linearization indeed predicts a center.

This system is <u>reversible</u>. This means it is invariant under the change of variables $x \to -x$, $t \to -t$:

$-dx/-dt = -y - (-x)^2 \Rightarrow dx/dt = -y - x^2$ ✓

$dy/-dt = -x \Rightarrow -dy/dt = -x \Rightarrow dy/dt = x$ ✓

This implies the origin is <u>truly</u> a center for the nonlinear system.