

Dummit & Foote (7.4) 24, 30, 31, 32

Remember, R is a ring with identity $1 \neq 0$.

7.4.24 Prove that in a Boolean ring every finitely generated ideal is principal.

7.4.30 Let I be an ideal of the commutative ring R with identity and define

$$\text{rad } I = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{Z}^+\}$$

called the *radical* of I . Prove that $\text{rad } I$ is an ideal containing I and that $(\text{rad } I)/I$ is the nilradical of the quotient ring R/I , i.e., $(\text{rad } I)/I = \mathfrak{N}(R/I)$ (cf. Exercise 29, Section 3).

7.4.31 An ideal I of the commutative ring R with identity is called a *radical ideal* if $\text{rad } I = I$.

- (a) Prove that every prime ideal of R is a radical ideal.
- (b) Let $n > 1$ be an integer. Prove that 0 is a radical ideal in $\mathbb{Z}/n\mathbb{Z}$ if and only if n is a product of distinct primes to the first power (i.e., n is square free). Deduce that (n) is a radical ideal of \mathbb{Z} if and only if n is a product of distinct primes in \mathbb{Z} .

7.4.32 Let I be an ideal of the commutative ring R with identity and define

$\text{Jac } I$ to be the intersection of all maximal ideals of R that contain I

where the convention is that $\text{Jac } R = R$. (If I is the zero ideal, $\text{Jac } 0$ is called the *Jacobson radical* of the ring R , so $\text{Jac } I$ is the preimage in R of the Jacobson radical of R/I .)

- (a) Prove that $\text{Jac } I$ is an ideal of R containing I .
- (b) Prove that $\text{rad } I \subseteq \text{Jac } I$, where $\text{rad } I$ is the radical of I defined in Exercise 30.
- (c) Let $n > 1$ be an integer. Describe $\text{Jac } n\mathbb{Z}$ in terms of the prime factorization of n .