Math 171 - Abstract Algebra I Section _
HW #14
10/28/10

Dummit & Foote (8.1) 3, 11, plus two outside problems.

Outside Problem # 1 Let R be an integral domain. We say a norm N on R is *monotonic* (disclaimer: I just made this terminology up) if $N(a) \leq N(ab)$ for any $a, b \in R$ with $ab \neq 0$. Let N be any norm which makes R into a Euclidean domain.

(a) Show that the function $N^* : R \to \mathbb{N}_{\geq 0}$ given by

$$N^*(a) = \min(\{N(ax) \mid x \in R, x \neq 0\}$$

is a monotonic norm on R.

(b) Show that N^* also makes R into a Euclidean domain. That is, the division algorithm in R works using N^* as the norm. (Hint: If I'm dividing a by b, consider $c \in R$ so that $N^*(b) = N(bc)$. What can I divide by bc that might be interesting?)

Note: Many books require a monotonic norm as part of the definition of a Euclidean domain, but this problem shows that requiring it is not strictly necessary. Moreover, monotonicity isn't directly needed to prove most things we want to know about Euclidean domains, which is why Dummit and Foote don't even mention it.

Outside Problem # 2 Let R be a Euclidean domain and assume its norm N is monotonic.

(a) Prove that a non-zero $a \in R$ is a unit if and only if $N(a) = N(1_R)$.

(b) Let $a, b \in R$. Prove that if d is a GCD of a and b, then $N(d) \geq N(d')$ for any d' that divides both a and b. Note: I believe if you pick your norm right, you can make the converse of part (b) true. But it's tricky....

8.1.3 Let R be a Euclidean domain (with a norm N that is not necessarily monotonic). Let m be the minimum integer in the set of norms of nonzero elements of R. Prove that every nonzero element of R of norm m is a unit. Deduce that a nonzero element of norm zero (if such an element exists) is a unit.

8.1.11 Let R be a commutative ring with 1 and let a and b be nonzero elements of R. A *least common multiple* of a and b is an element e of R such that

(i) $a|e$ and $b|e$, and

(ii) if $a|e'$ and $b|e'$ then $e|e'$.

(a) Prove that a least common multiple of a and b (if such exists) is a generator for the unique largest principal ideal contained in $(a) \cap (b)$.

(b) Deduce that any two nonzero elements in a Euclidean Domain have a least common multiple which is unique up to multiplication by a unit.

(c) Prove that in a Euclidean Domain the least common multiple of a and b is $\frac{ab}{(a,b)}$, where (a, b) is the greatest common divisor of a and b.