Math 171 - Abstract Algebra I Section _
HW # 15
11/1/10

Dummit & Foote (8.2) 3, 4, 5

8.2.3 Prove that a quotient of a P.I.D. by a prime ideal is again a P.I.D.

8.2.4 Let R be an integral domain. Prove that if the following two conditions hold then R is a Principal Ideal Domain:

(i) any two nonzero elements a and b in R have a greatest common divisor which can be written in the form $ra + sb$ for some $r, s \in R$, and

(ii) if $a_1, a_2, a_3, \ldots$ are nonzero elements of R such that $a_{i+1} | a_i$ for all i, then there is a positive integer N such that a_n is a unit times a_N for all $n \geq N$.

8.2.5 Let R be the quadratic integer ring $\mathbb{Z}[\sqrt{-5}]$. Define the ideals $I_2 = (2, 1+\sqrt{-5})$, $I_3 = (3, 2+\sqrt{-5})$, and $I_3' = (3, 2-\sqrt{-5})$.

(a) Prove that I_2, I_3, and I_3' are nonprincipal ideals in R. [Note that Example 2 following Proposition 1 proves this for I_3.]

(b) Prove that the product of two nonprincipal ideals can be principal by showing that I_2^2 is the principal ideal generated by 2, i.e., $I_2^2 = (2)$.

(c) Prove similarly that $I_2 I_3 = (1-\sqrt{-5})$ and $I_2 I_3' = (1+\sqrt{-5})$ are principal. Conclude that the principal ideal (6) is the product of 4 ideals: $(6) = I_2^2 I_3 I_3'$.