Math 171 - Abstract Algebra I Section __
HW #20
10/28/10

1. Let G be a group of order p^a and let N be a non-trivial normal subgroup of G. Show that $N \cap Z(G)$ is a non-trivial subgroup of G, where $Z(G)$ is the center of G. (In particular, $Z(G)$ is non-trivial.)

2.

(a.) Show that a Sylow p-subgroup of a group G is normal if and only if it is the unique Sylow p-subgroup of G.

(b.) Show that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing its order. (Hint: Counting elements might be helpful in one case.)

3. (University of Wisconsin Algebra Qualifying Exam, August '03 Problem 1) Let G be a group of order 504.

(a.) Show that G cannot be isomorphic to a subgroup of the alternating group A_7.

(b.) If G is simple, determine the number of Sylow 3-subgroups of G.

Note: For this problem, you will need the $n!$ theorem (Problem 2 from HW 20) and the fact that A_7 itself is simple. Also, if H is a subgroup of S_7, $H \cap A_7$ is often quite interesting.... This was one problem of five on this exam, and solving it completely earned a little less than one-third of the score needed to pass the exam.)