

Dummit & Foote (3.3) 3, 7, 8, 9

3.3.3 Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either

(i) $K \leq H$ or

(ii) $G = HK$ and $|K : K \cap H| = p$.

3.3.7 Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$. [Draw the lattice.]

3.3.8 Let p be a prime and let G be the group of p -power roots of 1 in \mathbb{C} (cf. Exercise 18, section 2.4). Prove that the map $z \mapsto z^p$ is a surjective homomorphism. Deduce that G is isomorphic to a proper quotient of itself.

3.3.9 Let p be a prime and let G be a group of order $p^a m$, where p does not divide m . Assume P is a subgroup of G of order p^a and N is a normal subgroup of G of order $p^b n$, where p does not divide n . Prove that $|P \cap N| = p^b$ and $|PN/N| = p^{a-b}$. (The subgroup P of G is called a *Sylow p -subgroup* of G . This exercise shows that the intersection of any Sylow p -subgroup of G with a normal subgroup N is a Sylow p -subgroup of N .)