Abstract Algebra
Lecture 20
Monday, 11/15/2010

1 Functors

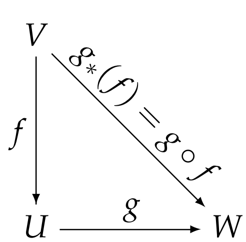
Recall: A **covariant functor** $F : \mathcal{A} \to \mathcal{B}$:

- $\forall A \in \text{Ob}(\mathcal{A}), F(A) \in \text{Ob}(\mathcal{B})$,
- $\forall \alpha : A \to A' \in \text{Mor}(\mathcal{A}), F(\alpha) : F(A) \to F(A') \in \text{Mor}(\mathcal{B})$ such that $F(\mathbb{1}_A) = \mathbb{1}_{F(A)}$ and $F(\alpha\beta) = F(\alpha)F(\beta)$ whenever $\alpha\beta$ is defined.

Ex: The Hom functor. Let $\mathcal{VS}_k$ be the category of vector spaces over the field k, with morphisms being linear maps (transformations). Let $\mathcal{A}b\mathcal{G}ps$ be the category of abelian groups. Fix a vector space V. Define the functor $\text{Hom}(V, \_) : \mathcal{VS}_k \to \mathcal{A}b\mathcal{G}ps$.
Claim: For a vector space $W \in \text{Ob}(\mathcal{VS}_k)$, then $\text{Hom}(V, W)$ is an abelian group. Let $\text{Hom}_k(V, W) = \{k\text{-linear maps } V \to W\}$. For $f, g \in \text{Hom}_k(V, W)$, define $(f + g)(v) = f(v) + g(v)$ for $v \in V$.
Check that this is a group. For $u, v \in V$, $a, b \in k$,

$$\begin{aligned}(f+g)(au+bv) &= f(au+bv) + g(au+bv)\\ &= af(u) + bf(v) + ag(u) + bg(v)\\ &= a(f+g)(u) + b(f+g)(v).\end{aligned}$$

So $(\text{Hom}(V, W), +)$ is closed. Does it have inverses? Yes: $-f \in \text{Hom}(V, W)$ for all $f \in \text{Hom}(V, W)$. Identity? Yes: the zero map $\mathbf{0}$. $\mathbf{0} : V \to W$ defined by $\mathbf{0}(v) = \vec{0} \in W$. We can see this because $(f + \mathbf{0})(v) = (\mathbf{0} + f)(v) = f(v) \forall f \in \text{Hom}(V, W)$.
So how does this Hom functor work on morphisms? Let U, W be vector spaces, $g : U \to W$ a linear map. Then $\text{Hom}(V, g) : \text{Hom}(V, U) \to \text{Hom}(V, W)$ and $\text{Hom}(V, g) = g_*$ defined by $g_* : f \mapsto g \circ f$.

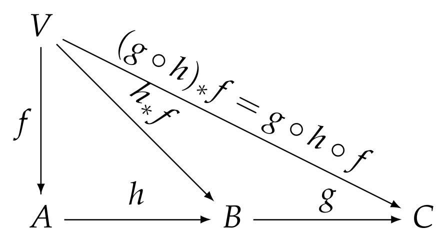


Check that this makes sense. Let f_1, f_2 be linear maps $f_1, f_2 : V \to U$. What is $g_*(f_1 + f_2)$?

Let $v \in V$. Then

$$\begin{aligned} g_*(f_1+f_2)(v) &= g \circ (f_1+f_2)(v) \\ &= f((f_1+f_2)(v)) \\ &= g(f_1(v)+f_2(v)) \\ &= g \circ f_1(v) + g \circ f_2(v) \\ &= g_* f_1(v) + g_* f_2(v). \end{aligned}$$

Now, let's check that this functor actually obeys the functor criteria.
What is $\text{Hom}(V, \mathbb{1}_W)$? $\mathbb{1}_W : W \to W$. So $(\mathbb{1}_W)_* : \text{Hom}(V,W) \to \text{Hom}(V,W)$. So $(\mathbb{1}_W)_* f = \mathbb{1}_W \circ f = f$. So, finally, $(\mathbb{1}_W)* = \mathbb{1}_{\text{Hom}(V,W)}$.
What about composition? Let $h : A \to B$ and $g : B \to C$. What is $(g \circ h)_*$? Well, $(g \circ h) : \text{Hom}(V,A) \to \text{Hom}(V,C)$. So what does a map f get mapped to?



Remark: Categories and Functors feel like the objects and morphisms of some category. Unfortunately, there is no class of all classes. So we can't do this.

Defn: A category $\mathcal{C}$ has **small Hom sets** in case $\text{Hom}(A,B)$ (all morphisms in $\mathcal{C}$ from A to B) is a set for all $A, B \in \text{Ob}(\mathcal{C})$.

Remark: When $\mathcal{C}$ has small Hom sets, there is a functor $\text{Hom}(A, \_) : \mathcal{C} \to \mathcal{S}ets$ for each object $A \in \text{Ob}(\mathcal{C})$.

Note: If $F : \mathcal{A} \to \mathcal{B}$ is a functor, and $G : \mathcal{B} \to \mathcal{C}$, then $G \circ F : \mathcal{A} \to \mathcal{C}$ is a functor.

Defn: For any category $\mathcal{C}$, the identity functor $\mathbb{1}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the identity functor on all objects and morphisms of $\mathcal{C}$.

Defn: $\mathcal{C}ats$ is the category of small categories and their functors.

Defn: When are categories "the same"? Two categories $\mathcal{A}$ and $\mathcal{B}$ are **isomorphic** if there exists functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ such that $F \circ G = \mathbb{1}_B$ and $G \circ F = \mathbb{1}_A$.

As it turns out, isomorphism of categories is incredibly strong, and leaves little wiggle room. Next time, we will look at "natural transformations" between categories, which lets us build up a notion of equivalence between categories.