Abstract Algebra
Lecture 21
Wednesday, 11/17/2010

1 Playing with Functors

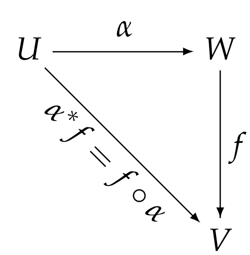
Defn: A **contravariant functor** $F : \mathcal{A} \to \mathcal{B}$ assigns:

- $A \in \text{Ob}(\mathcal{A}), F(A) \in \text{Ob}(\mathcal{B})$
- $\alpha \in \text{Mor}(\mathcal{A}), F(\alpha) \in \text{Mor}(\mathcal{B})$.

just like Covariant Functors. But this one reverses arrows: A Contravariant Functor must do these assignments such that:

- $F(\mathbb{1}_A) = \mathbb{1}_{F(A)}$
- if $\alpha : A_1 \to A_2$, then $F(\alpha) : F(A_2) \to F(A_1)$, and $F(\alpha\beta) = F(\beta)F(\alpha)$ when $\alpha\beta$ is defined.

Ex: The Hom functor! But a new one! Consider Hom : $\mathcal{VS} \to \mathcal{A}b\mathcal{G}ps$. Fix the vector space V and define Hom$(_, V)$. For all $W \in \mathcal{VS}$, Hom$(W, V) =$ Hom$(_, V)(W)$. Legt $\alpha : U \to W$ be a linear map. Then $Hom(\alpha, V) = \alpha^* :$ Hom$(W, V) \to$ Hom(U, V) defined by $f \mapsto \alpha^* f$.



1.1 Natural Transformation

Defn: Let $\mathcal{A}$ and $\mathcal{B}$ be categories, and $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A **natural transformation** $\tau : F \to G$ assigns object A in $\mathcal{A}$ a morphism $\tau_A : F(A) \to G(A)$ in $\mathcal{B}$ such that for all $\alpha : A \to A'$ in $\mathcal{A}$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\tau_A} & G(A) \\ {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle G(\alpha)} \\ F(A') & \xrightarrow{\tau_{A'}} & G(A') \end{array}$$

We also say τ is natural in A.

Defn: If each τ_A is an isomorphism, then τ_A^{-1} is also natural in A, and we call τ a **natural isomorphism**. We say $F \cong G$.

Defn: Two categories $\mathcal{A}$ and $\mathcal{B}$ are **equivalent** in case there exist functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ such that $F \circ G \cong \mathbb{1}_{\mathcal{B}}$ and $G \circ F \cong \mathbb{1}_{\mathcal{A}}$.

$$\begin{array}{ccccc} G \circ F(A) & \xrightarrow{\tau_A} & \mathbb{1}_{\mathcal{A}}(A) & = & A \\ {\scriptstyle G \circ F(\alpha)}\downarrow & & \downarrow{\scriptstyle \mathbb{1}_{\mathcal{A}}(\alpha)} & = & \downarrow{\scriptstyle \alpha} \\ G \circ F(A') & \xrightarrow{\tau_{A'}} & \mathbb{1}_{\mathcal{A}}(A') & = & A' \end{array}$$

2 Algebraic Geometry

Origins: Zero sets of polynomials.

Ex: $f(x,y,z) = x^2 + y^2 - z^2$, $(x,y,z) \in \mathbb{C}^3$. $V(f) = \{(a,b,c) \in \mathbb{C}^3 \mid f(a,b,c) = 0\}$.

Ex: $g(x,y,z) = x^3 + y^3 - z^3$. Whoops! Fermat! Can't be done.

2.1 Enumerative Geometry

Ex: Let l_1, l_2 be lines in $\mathbb{C}^2$. Count the number of intersections of the lines: $\#|l_1 \cap l_2| = 0, 1, \infty$.
Fact: for "general" points, there exists a unique conic through 5 points in $\mathbb{C}^2$. (Here, gen-

eral means no three colinear, etc.)
Consider: $Q(x_0, x_1, x_2, x_3, x_4) = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5$. How many lines live in the zero set of this polynomial, $V(Q)$? In 1881, a guy named Shubert showed there are exactly 2,875 of them. How about the number of conics live in $V(Q)$? In 1981, Sheldon Katz showed there are 609,250. But by the early 90s, all the rest were given. By string theory. But why does abstract algebra work on these? How does this make sense?

2.1.1 Algebraic Varieties

Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal. Then $V(I) = \{(a_1, \dots, a_n) \in k^n \mid f(a_1, \dots, a_n) = 0 \forall f \in I\}$. So we can associate a variety with an ideal. What about the other way around? Given a set $V \subseteq k^n$, let's look at the functions that vanish on those points: $I(V) = \{f \in k[x_1, \dots, x_n] \mid f(f) = 0 \forall p \in V\}$. And so, of course, we can have a category of varieties. In this category, the objects are things of the form $V(I)$. The morphisms are polynomials $\phi : V \to W \subseteq k^m$ defined by $p \mapsto (\phi_1(p), \dots, \phi_m(p))$ where each ϕ_i is a polynomial.

Ex: $V = V(x^2 + y^2 - 1) \subseteq \mathbb{C}^2$. And consider two example functions: $f : v \to \mathbb{C}^2$ defined by $f \mapsto 0$, and $g : V \to \mathbb{C}^2$ defined by $(x, y) \mapsto (x^2 + y^2 - a)x$. But both of these are essentially the same: they both map to 0! So we don't want the polynomials to be just any polynomials, we want them to be polynomials modulo those polynomials that vanish on our point set. So how do we do that? Quotients!

Defn: $k[V] = k[x_1, \dots, x_n]/I(V)$. For a morphism (i.e. polynomial map) $f : V \to W$, we can define $f^* : k[W] \to k[V]$. This works out to be an equivalence of categories:

$$\{\text{varieties of polynomial maps}\} \cong \{k[x_1, \dots, x_n]/I \text{ homomorphisms}\}.$$