Abstract Algebra
Lecture 1
Wednesday, 9/1/2010

1 General Information

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Webpage:
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Office Hours:
MON 3:30 - 4:30, TUE 4:00 - 5:00 in Olin 1267.
Prof Davis's office is at Olin B160.

Grades:
Three exams, Homework, and the max of all of the above, 20% each.

Exam dates (takehome):
WED Sept 29, WED Nov 3, WED Dec 8.

Homework:
Due twice a week on MON and THUR.

Rewrites:
Always do rewrites for homeworks!

Note: Get the textbook!

2 Introduction

Basic object of study: The Set.
Different branches of math equip sets with different additional structures.

Defn: Algebra is the study of sets with additional structure.

Consider $\mathbb{R}^2$, the real plane. We can look at lines in the plane, and study their angles. That is geometry. Topology treats the plane as a bendable substance that can stretch, but not rip. Or we can consider points in the plane, and want them to be able to talk to each other. This is algebra.

2.1 Set Theory

Defn: Cartesian product
The Cartesian Product of two sets A, B is the set

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

Defn: Map
A map $A \longrightarrow B$ of sets is a subset $S \subseteq A \times B$ such that $\forall a \in A, \exists! b \in B$ such that $(a, b \in S)$. We say $a \mapsto b$ (find the right notation for this).

Defn: Binary operation
A binary operation $*$ on a set A is a map $A \times A \longrightarrow A$. In other words, $(a_1, a_2) \mapsto a_1 * a_2 = *(a_1, a_2)$.

Ex: Let $A = \mathbb{R}$, $* = +$. Then $*$ is a map $(a, b) \mapsto a + b$.

Defn: Closure
Let $B \subseteq A$, and $*$ a binary operation on A. Then $*$ is closed on B if $\forall b_1, b_2 \in B, b_1 * b_2 \in B$.

Ex: Let $A = \mathbb{R}, B = \mathbb{N}$ (not including 0). Then the binary operation $* = -$ is not closed on B.

Defn: Associativity
A binary operation $*$ on A is associative if $\forall a_1, a_2, a_3 \in A$, $a * (a_2 * a_3) = (a_1 * a_2) * a_3$.

Defn: Identity
An element $e \in A$ is an identity element of $*$ on A if $\forall a \in A, e * a = a * e = a$.

Defn: Inverse
An element $a \in A$ has an inverse under the binary operation $*$ if $\exists b \in A$ such that $a * b = b * a = e$, the identity element.

2.2 Groups

Defn: Group
A Group is a pair $(G, *)$ of a set and a binary operation such that $*$ is associative, $(G, *)$ has an identity element, and every $g \in G$ has an inverse in G.

Note: The binary operation is closed on G - this comes for free. If it were not closed, then it would not be a true binary operation. Recall that is has to map back onto G itself.

Ex: Groups
$(\mathbb{R}, +)$ is a group.
$(\mathbb{Z}, \cdot)$ is not a group.
$(\mathbb{R}, \cdot)$ is not a group.
$(\mathbb{R} - 0, \cdot)$ is a group.
$(\{1, -1\}, \cdot)$ is a group.
$(\mathbb{N}, +)$ is not a group.
$(\mathbb{Z}, +)$ is a group.
Modular arithmetic: $Z_n = \{0, 1, \ldots, n-1\}$, $a * b = c \Leftrightarrow n|(a+b) - c$ is a group.

2.3 Theorems

Theorem: Let G be a group (under some implied binary operation $*$).

1. $e \in G$ is unique
2. $\forall a \in G, \exists!\, a^{-1} \in G$
3. $\forall a \in G, (a^{-1})^{-1} = a$
4. $(ab)^{-1} = b^{-1}a^{-1} \forall a, b \in G$

Proof:

1. Suppose e, e' are identity elements in G. Then, $\forall a \in G$, $a * e = e * a = a$, and $a * e' = e' * a = a$. Thus, $e = e * e' = e'$.
2. Suppose $b * a = a * b = e$. Then $b = b * (a * a^{-1}) = (b * a) * a^{-1} = a^{-1}$.
3. $a * a^{-1} = e$, and $a^{-1} * (a^{-1})^{-1} = e$, so $a = (a^{-1})^{-1}$.
4. $(a * b) * (b^{-1} * a^{-a}) = abb^{-1}a^{-1} = aa^{-1} = e$ etc.

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