

Recall: Group isomorphism is our notion of equivalence or congruence, notated $G \cong H$.

1 Quotient Groups

1.1 Quotient Sets

Our goal is to take the Quotient of a group G by a subgroup N . This builds on the notion of a quotient set.

Recall: An equivalence relation \mathcal{E} on a set S is a binary relation (i.e. $\mathcal{E} \subseteq S \times S$) which is reflexive, symmetric, and transitive. We denote an equivalence relation by

$$(a, b) \in \mathcal{E} \iff a \sim b.$$

Reflexive: $\forall a \in S, a \sim a$

Symmetric: for all $a, b \in S, a \sim b \rightarrow b \sim a$

Transitive: $a \sim b, b \sim c \rightarrow a \sim c$.

Defn: Let \mathcal{E} be an equivalence relation on a set S . For any $a \in S$, the equivalence class of $a \pmod{\mathcal{E}}$ is given by

$$[a] = \{b \in S \mid a \sim b\}.$$

Defn: Let \mathcal{E} be an equivalence relation on S . The **quotient set** of S by \mathcal{E} is the set of all equivalence classes modulo \mathcal{E} .

$$S/\mathcal{E} = \{[a] \mid a \in S\}.$$

Ex: Let $S = \mathbb{Z}$, and let $(a, b) \in \mathcal{E} \subseteq \mathbb{Z} \times \mathbb{Z} \iff a \equiv b \pmod{3}$.

So, $\mathbb{Z}/\mathcal{E} = \{[0], [1], [2]\}$, where $[0] = 0, 3, 6, 9, \dots$, $[1] = 1, 4, 7, \dots$, etc.

Defn: The quotient set S/\mathcal{E} comes equipped with a **projection map** $\pi : S \rightarrow S/\mathcal{E}$ defined by $\pi(a) \mapsto [a]$.

1.2 Quotient Groups

Defn: A group G' is a quotient group of the group G if

- (1) G' is a quotient set of G , i.e. $G' = G/\mathcal{E}$ for some equivalence relation on G .
- (2) The projection map $\pi : G \rightarrow G'$ is a group homomorphism.

Note: All quotient groups are constructed from subgroups.

Let $\phi : G \rightarrow G'$ be a homomorphism. Then ϕ induces an equivalence relation on G :

$$x \sim y \iff \phi(x) = \phi(y) \in G'.$$

We know that $\phi(xy) = \phi(x)\phi(y)$. So notice that $\phi(x)(\phi(y))^{-1} = 1_{G'}$. Then $\phi(x)\phi(y^{-1}) = 1$. So $\phi(xy^{-1}) = 1$. Thus, the equivalence relation is determined by

$$K = \{x \in G \mid \phi(x) = 1\}.$$

In other words,

$$x \in y \iff xy^{-1} \in K \iff x^{-1}y \in K \iff y^{-1}x \in K \iff yx^{-1} \in K.$$

We say that

$$K = \ker \phi,$$

and $\ker \phi \leq G$. K is the **Kernel**. It is where things go to die. Everything in the Kernel gets mapped to 0. It completely defines this equivalence relation.

Note: For any quotient group G' of G ,

$$x \sim y \iff [x] = [y],$$

which means

$$\pi(x) = \pi(y),$$

which is true if and only if

$$xy^{-1} \in \ker \pi.$$

Note: Let A, B be subgroups of G . Then $AB = \{ab \mid a \in A, b \in B\}$. Similarly, for any $a_0 \in A$, $a_0B = \{a_0b \mid b \in B\}$, and $Ba_0 = \{ba_0 \mid b \in B\}$.

Defn: The **right coset** of a subgroup H of a group G by the element x is the set Hx . Similarly, the **left coset** is the set xH .

Theorem: Let $\phi : G \rightarrow G'$ be a homomorphism, and $K = \ker \phi$. Then $xKx^{-1} \subseteq K$ for all $x \in G$.

Proof: $xKx^{-1} = \{xkx^{-1} \mid k \in K\}$. We want to show that $xkx^{-1} \in K$ for all $k \in K$. Let $k \in K$. Then $\phi(xkx^{-1}) = \phi(x)\phi(k)\phi(x^{-1}) = \phi(x)(1)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(1) = 1$. QED. ■

Defn: A **normal subgroup** of a group G is a subgroup $N \leq G$ such that $xNx^{-1} \subseteq N$ for all $x \in G$.

Theorem: Let $N \leq G$. Then the following are equivalent.

1. $N \trianglelefteq G$ ($xNx^{-1} \subseteq N$)
2. $xNx^{-1} = N$
3. $xN = Nx$
4. $\forall x, y \in G, xy^{-1} \in N \iff y^{-1}x \in N$.

Proof:

- 1 \rightarrow 2: Let $n \in N, x \in G$ is given, so $x^{-1} \in G$. Since $N \trianglelefteq G, x^{-1}Nx \subseteq N$. So $x^{-1}nx \in N$. Then $x(x^{-1}nx)x^{-1} \in xNx^{-1}$, so $n \in xNx^{-1}$.
- 2 \rightarrow 3: Left as an exercise for the reader.
- 3 \rightarrow 4: Left as an exercise for the reader. ■

Theorem: When $N \trianglelefteq G$, the (right) cosets of N by elements of G form a group $G/N = \{Ng \mid g \in G\}$ under the operation $Ng_1 \cdot Ng_2 = Ng_1g_2$. And, the map $\pi : G \rightarrow G/N$ defined by $x \mapsto Nx$ is a homomorphism onto G/N . And, $\ker \pi = N$.

Ex: Let $G = (\mathbb{Z}, +)$, and $N = \{0, \pm 3, \pm 6, \pm 9, \dots\} = 3\mathbb{Z}$. Note that $Nx = N + x = 3\mathbb{Z} + x$, so $xNx^{-1} = x + 3\mathbb{Z} + x^{-1} = x - x + 3\mathbb{Z} = 3\mathbb{Z}$. Because this group is abelian, every subgroup is a normal subgroup. The set of cosets $\mathbb{Z}/3\mathbb{Z} = \{3\mathbb{Z}, 3\mathbb{Z} + 1, 3\mathbb{Z} + 2\}$. So, finally, $\mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}_3$.