Abstract Algebra
Lecture 8
Monday, 9/27/2010

1 Isomorphism Theorems, Continued

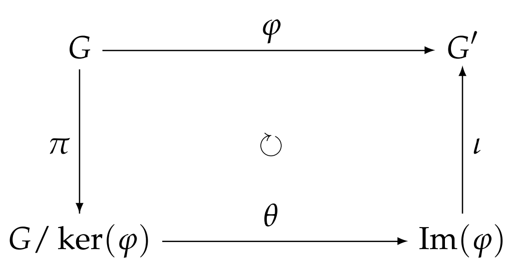
1.1 The First Isomorphism Theorem

Recall: The First Isomorphism Theorem or The Homomorphism Theorem:
Let $\varphi : G \to G'$ be a homomorphism of groups. Then

$$G/\ker(\varphi) \cong \mathrm{Im}(\varphi).$$

Note that this implies a) $\ker(\varphi)$ is a normal subgroup of G, and b) $\mathrm{Im}(\varphi)$ is a group.

Defn: In fact, there exists an isomorphism $\theta : G/\ker(\varphi) \xrightarrow{\sim} \mathrm{Im}(\varphi)$ such that this **diagram commutes**:



$$\varphi = \iota \circ \theta \circ \pi$$

In other words, $\forall x \in G$, $\varphi(x) = \iota(\theta(\pi(x)))$.

Ex: Let G be a group. Let $a \in G$ be any element. If a has infinite order, then $\langle a \rangle \cong \mathbb{Z}$ (under addition, of course). If a has finite order n, then $\langle a \rangle \cong \mathbb{Z}_n$.

Proof: Consider the map $\pi : \mathbb{Z} \to G$ defined by $\pi(k) = a^k$. If a has infinite order, i.e. there does not exist $k \neq 0$ such that $a^k = 1$, then $\ker(\pi) = \{0\}$. Note that $\mathrm{Im}(\pi) = \{a^k \mid k \in \mathbb{Z}\} = \langle a \rangle$. But by the First Isomorphism Theorem, we know that $\mathbb{Z}/\ker(\pi) \cong \mathrm{Im}(\pi)$, so $\mathbb{Z} \cong \langle a \rangle$ (because $\mathbb{Z}/\{0\} \cong \mathbb{Z}$, and $\mathrm{Im}(\pi) = \langle a \rangle$). If a has finite order, i.e. $|a| = n$. Inspect $\ker(\pi)$. We have $\ker(\pi) = \{k \in \mathbb{Z} \mid a^k = 1 \in G\} = n\mathbb{Z}$. So, by the First Isomorphism Theorem, $\mathbb{Z}/\ker(\pi) = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n \cong \langle a \rangle$. ∎

Ex: Consider the unit circle in the complex plane (S^1). Recall that we can "coil" the real line along this unit circle using the sine and cosine functions. Let us make this relationship more concrete. Consider the map $\varphi : \mathbb{R} \to S^1$ given by $t \mapsto e^{it}$. Then $\text{Im}(\varphi) = S^1$, and $\ker(\varphi) = \{t \in \mathbb{R} \mid \varphi(t) = 1\} = \langle 2\pi \rangle = 2\pi\mathbb{Z}$. By the First Isomorphism Theorem, $\mathbb{R}/\langle 2\pi \rangle \cong S^1$.

1.2 The Third Isomorphism Theorem

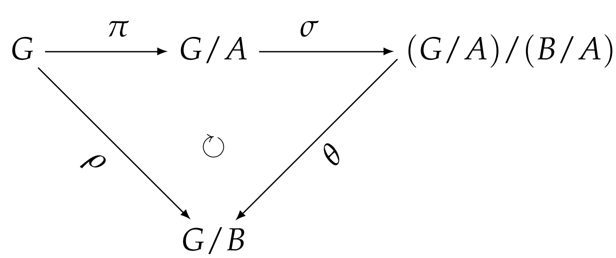
Motivation: **Theorem:** Let $N \trianglelefteq G$. Every subgroup of G/N is of the form H/N, for some unique subgroup $H \leq G$ containing N. Furthermore, $H/N \trianglelefteq G/N$ if and only if $H \trianglelefteq G$.

Remark: Let $\varphi : G \to G'$ be a group homomorphism. Let $H' \leq G'$. Then $\varphi^{-1}(H') \leq G$. Also, $H \leq G$, then $\varphi(H) \leq G'$. "This sets up a bijection between subgroups on the one hand side, and subgroups on the other hand side." Consider $\pi : G \to G/N$. Let $A \leq G/N$. Study $\pi^{-1}(A) \leq G$. $N = 1 \in G$. $\pi^{-1}(1) \subseteq \pi^{-1}(A)$ because A is a group. So $\pi^{-1}(N) \subseteq \pi^{-1}(A)$. $\ker(\pi) \subseteq \pi^{-1}(A)$. Thus $N \subseteq \pi^{-1}(A)$.

Theorem: Let G be a group, $A \trianglelefteq G, B \trianglelefteq G$. If $A \subseteq B$, then $A \trianglelefteq B$, and $B/A \trianglelefteq G/A$, and

$$(G/A)/(B/A) \cong G/B.$$

In fact, there exists an isomorphism $\theta : (G/A)/(B/A) \xrightarrow{\sim} G/B$ such that this diagram commutes:



where π, ρ, σ are projections. In other words, $\rho = \theta \circ \sigma \circ \pi$.

Proof: Define $\omega : G/A \to G/B$ by $Ax \mapsto Bx$. By the First Isomorphism Theorem, $B/A \trianglelefteq G/A$ by inspection of the kernel. $\ker(\omega) = \{Ax \in G/A \mid Bx = B\} = \{Ax \in G/A \mid x \in B\} = B/A$. $(G/A)/\ker(\omega) \cong \text{Im}(\omega)$, so $(G/A)/(B/A) \cong G/B$. ∎

1.3 The Second Isomorphism Theorem

Theorem: Let G be a group, $A \leq G$, $N \trianglelefteq G$. Then $AN \leq G$, and $N \trianglelefteq AN$, and $A \cap N \trianglelefteq A$, and $AN/N \cong A/(A \cap N)$. In fact, there exists an ismorphism $\theta : AN/N \to A/(A \cap N)$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\quad \iota \quad} & AN \\ {\scriptstyle \pi}\downarrow & \circlearrowright & \downarrow{\scriptstyle \rho} \\ A/(A \cap N) & \xleftarrow[\quad \theta \quad]{} & AN/N \end{array}$$

Ex: Consider $GL_n(\mathbb{C})$, the group of invertible linear transformations from $\mathbb{C}^n \to \mathbb{C}^n$. Consider $SL_n(\mathbb{C})$, maps with determinant 1. Then $GL_n(\mathbb{C})/SL_n(\mathbb{C}) \cong \mathbb{C}^\times$.