Syzygies and the Smoothable Component of the Hilbert Scheme of Points

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Overview/Notation

Overview

The Hilbert scheme of n points in $\mathbb{P}^d$ has a "main component", whose generic element parametrizes a smooth 0-dimensional subscheme (i.e. a collection of n disjoint points in $\mathbb{P}^d$). Studying this component is equivalent to understanding which 0-dimensional schemes $\Gamma \subseteq \mathbb{P}^d$ deform to a disjoint union of points. **Our main result is the construction of a graded module M_Γ whose syzygies detect smoothability of $\Gamma \subseteq \mathbb{P}^d$.**

The smoothable component

We work over an algebraically closed field of characteristic not 2 or 3. The Hilbert scheme of n points in $\mathbb{P}^d$ is denoted by $\mathrm{Hilb}^n(\mathbb{P}^d)$. The **smoothable component** of $\mathrm{Hilb}^n(\mathbb{P}^d)$, denoted $\mathrm{Smbl}^n(\mathbb{P}^d)$, is the closure of the locus of smooth 0-dimensional subschemes of $\mathbb{P}^d$. $\mathrm{Smbl}^n(\mathbb{P}^d)$ parametrizes 0-dimensional subschemes of $\mathbb{P}^d$ which deform to a union of n distinct points.

Notation for Γ and M_Γ

We define a graded module M_Γ whose syzygies of M_Γ encode deformation theoretic information about Γ. As a vector space

$$(M_\Gamma)_i := \begin{cases} H^0(\Gamma, \mathcal{I}_\Gamma) & i = 0 \\ H^1(\Gamma, \mathcal{O}_\Gamma(1))^* & i = 1 \\ 0 & i \notin \{0,1\}. \end{cases}$$

Let $V := H^1(\Gamma, \mathcal{I}_\Gamma(1))^*$. We make M_Γ into a $\mathrm{Sym}_\bullet(V)$-module via the adjoint of the multiplication map. We set $e := \dim V - 1$.

Measuring extra syzygies with $b_i(\Gamma)$

We set $b_i(\Gamma) \in \mathbb{Z}_{\geq 0}$ as the number of "extra" i'th syzygies of M_Γ in degree i. Specifically, we expect $\dim_k \mathrm{Tor}^i(M_\Gamma, k)_i$ to equal $(d+1)(e-1)$, and we define

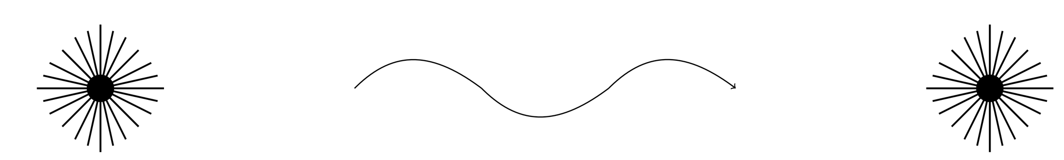
$$b_i(\Gamma) := \dim_k \mathrm{Tor}^i(M_\Gamma, k)_i - (d+1)(e-1).$$

Summary of Main Results

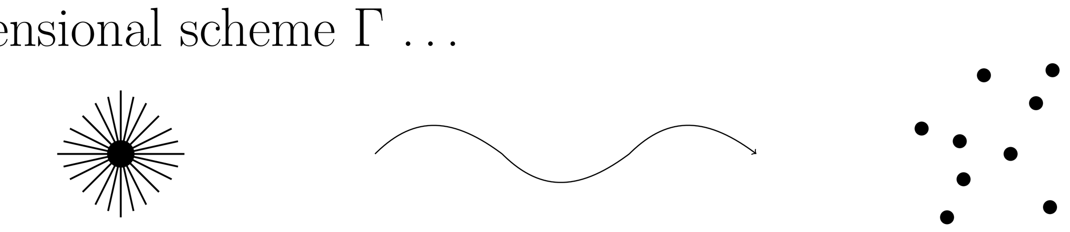
Example: Extra syzygies of M_Γ correspond to deformations of Γ

Let $\Gamma \subseteq \mathbb{P}^5$ be a 0-dimensional scheme of degree 9 supported at a single point. Since Γ is a nonreduced scheme, we draw Γ as a point with tangent vectors.

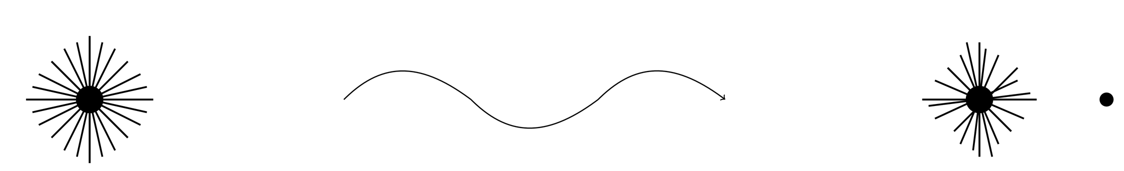
If $b_1(\Gamma) = 0$ and $b_2(\Gamma) = 0$, then the 0-dimensional scheme Γ … | … only deforms to other schemes supported at a single point.



If $b_2(\Gamma) > 0$ then the 0-dimensional scheme Γ … | … deforms to 9 distinct points.



If $b_1(\Gamma) > 0$ then… | … we can peel a single point off of Γ.



The inequalities involving the $b_i(\Gamma)$ lead to matrix equations for the intersections between components of the $\mathrm{Hilb}^9(\mathbb{P}^5)$. These matrix equations cut out $\mathrm{Smbl}^9(\mathbb{P}^5) \subseteq \mathrm{Hilb}^9(\mathbb{P}^5)$.

Theorem 1: Necessary Conditions for Smoothability

If Γ is smoothable, then M_Γ has extra syzygies. Namely,

$$b_1(\Gamma) \geq \binom{d-2}{2}. \qquad (*)$$

Further, for $j = 2, \ldots, \lfloor \frac{e}{2} \rfloor$,

$$b_j(\Gamma) \geq (\deg \Gamma - 1)\binom{e-1}{j-1} - e\binom{e}{j}. \qquad (**)$$

Comparison with previous results

- Obstructions to smoothability were constructed in [3] and [4] based on tangent space conditions. Our obstructions from Thm. 1 are generally sharper than these obstructions.
- Thm. 2 is essentially unique in the literature. It covers all known nontrivial cases of sufficient conditions for smoothability. (The case $d = 4$ first appeared in [2].)

Theorem 2: Sufficient Conditions for Smoothability

If $e = 3$ and $d = \mathrm{edim}(\Gamma) \leq 8$, then (*) and (**) are sufficient conditions for smoothability.

Corollary: A Divisorial Contraction

Let $n \geq 15, d \geq 11$. Consider the Chow morphism

$$\pi : \mathrm{Smbl}^n(\mathbb{P}^d) \to \mathrm{Sym}^n(\mathbb{P}^d)$$

which sends a 0-dimensional subscheme $\Gamma \subseteq \mathbb{P}^d$ to its corresponding cycle. There exists a divisor $Z \subseteq \mathrm{Smbl}^n(\mathbb{P}^d)$ such that $\pi(Z)$ equals the closed orbit of $\mathrm{Sym}^n(\mathbb{P}^d)$.

Remark: This corollary answers a question of Iarrobino regarding the largest possible dimension of a smoothable component of the punctual Hilbert scheme.

Motivation/Applications

Motivation for studying $\mathrm{Smbl}^n(\mathbb{P}^d)$

1. Fibers of finite maps: The fibers of a finite, flat, generically étale map $f : X \to Y$ are all *smoothable* 0-dimensional schemes. Conversely, the smoothable 0-dimensional schemes are precisely those schemes which can arise as the fiber of a finite, flat, generically étale map. This is relevant for the study of moduli spaces of finite maps (current work of Erman and Wood).
2. Combinatorics: Haiman's investigation of the geometry of $\mathrm{Hilb}^n(\mathbb{P}^2)$ led to combinatorial results about symmetric functions. He has conjectured that generalizations of his results may be related to an understanding of the geometry of Smbl^n_d for $d > 2$. (Current work of Kyungyong Lee).
3. Gale duality: Recent work of Eisenbud, Erman and Velasco illustrates a connection between Gale duality of points and smoothability of 0-dimensional schemes.

Further work

- Kyungyong Lee has shown that $\mathrm{Smbl}^9(\mathbb{P}^8)$ has singularities which are not Cohen-Macaulay.
- Defining equations for $\mathrm{Smbl}^n(\mathbb{P}^d) \subseteq \mathrm{Hilb}^n(\mathbb{P}^d)$ are related to defining equations for the secant varieties of $\mathbb{P}^m \times \mathbb{P}^n$ embedded by $\mathcal{O}(1,2)$. This is the subject of current work of Cartwright, Erman, and Oeding.

References

[1] Cartwright, D.; Erman, D.; Velasco, M.; Viray, B., *Hilbert schemes of 8 points.* Algebra Number Theory 3 (2009), no. 7, 763–795.

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[4] Shafarevich, I.R., *Deformations of commutative algebras of class 2.* Algebra i Analiz 2 (1990), no. 6, 178–196.