Varieties Fibered by Good Minimal Models

Ching-Jui Lai

Department of Mathematics, University of Utah

Aim

For a given algebraic variety X, we want to find a "good" representative in its birational class, and use this to study the geometry.

Baby Example

Let X be a smooth projective surface.

- Suppose that X contains a (-1)-curve C, i.e. $C \cong \mathbb{P}^1$ with $C^2 = -1$. Then by Castelnuovo's Theorem, one can contract this (-1)-curve by a birational morphism $X \to X'$ to a smooth surface X'.
- Since this procedure can happen only finitely many times, we end up with a birational morphism $X \to X_{\min}$ such that $X_{\min}$ is a smooth projective surface with $\kappa(X_{\min}) = \kappa(X)$ and contains no (-1)-curves.
- If $\kappa(X_{\min}) = \kappa(X) \geq 0 \Rightarrow K_{X_{\min}}$ nef. (Then we study the Enrique's classification.)

Remark:

- $\kappa(X) \geq 0 \Rightarrow$ the linear series $|mK_{X_{\min}}|$ is *base point free* for sufficiently divisible $m > 0$. (**Abundance**).
- If $\kappa(X) \geq 0$, then the output $X_{\min}$ is *uniquely* determined by the above algeorithm (up to isomorphism).

Minimal Model Program (MMP)

Goal: Want to extend the above process to higher dimensional varieties.

We start with the generalization of $X_{\min}$.

Definition A *minimal model* (**MM**) for a quasi-projective variety (normal, $\mathbb{Q}$-factorial with "mild" singularities) X is a birational map $\phi : X \dashrightarrow X'$ such that the following hold:

- X' is quasi-projective, normal and $\mathbb{Q}$-factorial,
- ϕ extracts no divisors,
- $K_{X'}$ is nef, and
- $a(F, X) < a(F, X')$ for any ϕ-exceptional divisors F on X. ($\bigstar$)

A minimal model X' of X is called "*good*" if $K_{X'}$ is semiample, i.e. abundance holds on X'.

Remark

- The condition ($\bigstar$) ensures that the singularities of X' are "better" than those of X.
- The setting for MMP can be generalized to pairs (X, Δ) relative over a normal quasi-projective variety U, i.e. relative log-MMP.
- In dim ≥ 3, **MM**s are *not* unique due to the exsistence of **FLOP**s!!!

Main Conjectures

Existence of MM: For a given variety X with $\kappa(X) \geq 0$, there exists a minimal model $\phi : X \dashrightarrow X'$.

Abundance: For any minimal model X' of X, the linear system $|mK_{X'}|$ is base point free for $m > 0$ sufficiently divisible.

Well-Known Facts

These conjectures are established in several cases:

- $\dim X \leq 3$ by Y. Kawamata, S. Mori, and others;
- Varieties of general type by [BCHM06].

From [BCHM], we also have the following very useful colloraries.

- For a klt pair (X, Δ), the canonical ring
$$R(X, K_X + \Delta) = \oplus H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$$
is finitely generated over $\mathbb{C}$.
- Finiteness of **MM**s for "big" pairs (cf. uniqueness of minimal model in dim 2).
- Any two minimal models can be connected by a sequence of flops, (cf. [Kawamata]).
- Assume that (X, Δ) has a good minimal model, then any MMP of (X, Δ) with scaling by an ample divisor terminates, (cf. [Lai]).

Question: MM's for Fiber Spaces

Let $f : X \to Y$ be an algebraic fiber space, i.e. a proper surjective morphism of normal (positive dimensional) varieties with connected fibers. Assume the existence of good minimal model for the general fiber F and the base Y, then does X have a good minimal model?

Main Results I

Theorem A Let X be a projective variety, normal and $\mathbb{Q}$-factorial with at most terminal singularities. Suppose $\kappa(X) \geq 0$ and the general fiber F of the Iitaka fibration has a good minimal model, then so does X.

Theorem B Let X be a projective variety, normal and $\mathbb{Q}$-factorial with at most terminal singularities. Assume $\kappa(X) = 0$, then (by a result of Kawamata) the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is an algebraic fiber space. If the general fiber F of α has a good minimal model, then so does X.

Sketch of Proof

Let $f : X \to Y$ be an algebraic fiber space with general fiber F. We establish our results by the following steps:

- Assume that F has a good minimal model, then by running MMP with scaling over Y, we get a birational map $X \dashrightarrow X'$ over Y such that the new general fiber F' of $f' : X' \to Y$ is a good minimal model of F.
- We can assume $\mathbf{B}_{-}(K_{X'}/Y)$ contains no divisorial components.
- There is an effective divisor Γ such that $K_{X'} \sim_{\mathbb{Q}} (f')^*M + \Gamma$ with M an ample divisor divisor on Y (in Theorem A) or $M = 0$ (in Theorem B).
- Suppose that $\Gamma \neq 0$.
- In both cases (Theorem A and B) $\kappa(F) = 0$, hence $K_{F'}$ is torsion and Γ can't dominant Y. The image $f'(\Gamma)$ in Y is a proper closed subset.
- Either in Theorem A or B, Γ can only be exceptional or doesn't contain all those divisorial components of $(f')^{-1}(f'(\Gamma))$ which maps down to a codimension one point in Y.
- Under the above situation, we can show (by a surface intersection computation) that Γ contains a divisorial component belonging to $\mathbf{B}_{-}(K_{X'}/Y)$. A contradiction!
- We conclude that $K_{X'}$ is semiample and hence X' is a good minimal model of X.

Main Result II

Theorem C Let X be a projective variety, normal and $\mathbb{Q}$-factorial with at most terminal singularities. If the general fiber F of the Albanese morphism $\alpha : X \to \mathrm{Alb}(X)$ has $\kappa(F) \geq 0$ with $\dim F \leq 3$, then X has a good minimal model.

Proof

The proof is based on an interesting nonvanishing result.

Nonvanishing Theorem Let X be an irregular smooth projective variety with Albanese morphism $\alpha : X \to Alb(X)$. If the general fiber F of α has $\kappa(F) \geq 0$, then $\kappa(X) \geq 0$. (The proof mainly uses the *generic vanishing theorems* of abelian varieties.)

- In case $\kappa(X) = 0$, this is Theorem B.
- If $\kappa(X) > 0$, we consider the Iitaka fibration of X and show that the general fiber of the Iitaka fibration has a good minimal model.

Iitaka's Conjecture C

For an algebraic fiber space $f : X \to Y$ with general fiber F, the Iitaka's conjecture C asks if $\kappa(X) \geq \kappa(F) + \kappa(Y)$. Combined with a result by Kawamata, which asserts that Iitaka's conjecture C holds if F has a good minimal model, we prove that

Corollary If the Iitaka codimension of F is no more than three, then Iitaka's conjecture C holds.

Acknowledgements

The author is grateful to his advisor, Professor Christopher Hacon, for many useful suggestions, discussions, and his generosity.

References

[1] C. Birkar, P. Cascini, C. Hacon, and J. McKernan: *Existence of minimal models for varieties of log general type*, 2006.

[2] J. Kollár and S. Mori: *Birational geometry of algebraic varieties*, Cambridge University Press, 1998.

[3] C. Lai: *Varieties fibered by minimal models.* arXiv:0912.3012v2