Moduli of PT-stable complexes

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Introduction

The study of derived categories has been motivated by the classification of algebraic varieties and, more recently, mirror symmetry. Just as one can study the moduli spaces of sheaves on a variety X, so one can study the moduli spaces of objects in the derived category $D^b(X)$ of coherent sheaves on X.

In my work, I lift the technique of semistable reduction for sheaves to the case of derived objects. Using this, I show that for a smooth projective three-fold X, the moduli space of PT-stable objects in $D^b(X)$ is of finite type (i.e. bounded) and satisfies the valuative criterion for properness.

What is PT-stability?

Let X be a smooth projective three-fold. PT-stability is defined for objects in the heart

$$\mathcal{A}^p(X) = \langle \mathrm{Coh}_{\leq 1}(X), \mathrm{Coh}_{\geq 2}(X)[1] \rangle \subset D^b(X),$$

which is an abelian category.

A complex $E \in D^b(X)$ is in $\mathcal{A}^p$ if and only if:

- $\dim \operatorname{supp} H^0(E)$ is at most 1-dimensional;
- $H^{-1}(E)$ has no subsheaves with 0- or 1-dimensional support;
- $H^i(E) = 0$ for all $i \neq 0, -1$.

Using a complexified version of the Hilbert polynomial

$$Z(E)(m) := \sum_{d=0}^{3} \int_X \rho_d H^d ch(E) m^d,$$

(where the ρ_d are complex numbers satisfying the configuration as in the Figure on the right), we define a quantity $\phi(E)(m) \in (0,1]$ for $m \gg 0$ by the relation

$$Z(E)(m) \in \mathbb{R}_{>0} e^{i\pi\phi(E)(m)}.$$

Definition of 'PT-semistable'

Using $\phi(E)(m)$, we say a nonzero object $E \in \mathcal{A}^p$ is PT-semistable if for any nonzero subobject $G \hookrightarrow E$ in $\mathcal{A}^p$, we have

$$\phi(G)(m) \leq \phi(E)(m) \text{ for } m \gg 0.$$

Results

Let X be a smooth projective three-fold over a field k of characteristic 0, and (R, π) be a discrete valuation ring over k with field of fractions K.

Proposition (boundedness). The set of all PT-semistable objects E in the heart $\mathcal{A}^p \subset D^b(X)$ with fixed $ch(E)$ is bounded.

Theorem (completeness of heart). A K-flat family of objects in the heart $\mathcal{A}^p$ can be extended to an R-flat family of objects in the heart $\mathcal{A}^p$.

Proposition (separatedness). Let $E_1, E_2 \in D^b(X \otimes_k R)$ be such that their generic fibers are isomorphic in $D^b(X \otimes_k K)$. If both their central fibers (derived restrictions to $D^b(X \otimes_k R/\pi)$) are PT-semistable, and one of them is PT-stable, then they are isomorphic.

Theorem (properness). Given a K-flat family of PT-semistable objects in $\mathcal{A}^p$ with $ch_0 \neq 0$ and ch_0, ch_1 coprime, it can be extended to an R-flat family of PT-semistable objects in $\mathcal{A}^p$.

Proposition (openness). Let S be a Noetherian scheme over k, and $E \in D^b(X \times_k S)$ be an S-flat family of objects in the heart $\mathcal{A}^p$ whose fibers have $ch_0 \neq 0$ and ch_0, ch_1 coprime. Suppose $s_0 \in S$ is a point such that $E|_{s_0}$ is PT-semistable. Then there is an open set $U \subseteq S$ containing s_0 such that for all $s \in U$, the fiber $E|_s$ is PT-semistable.

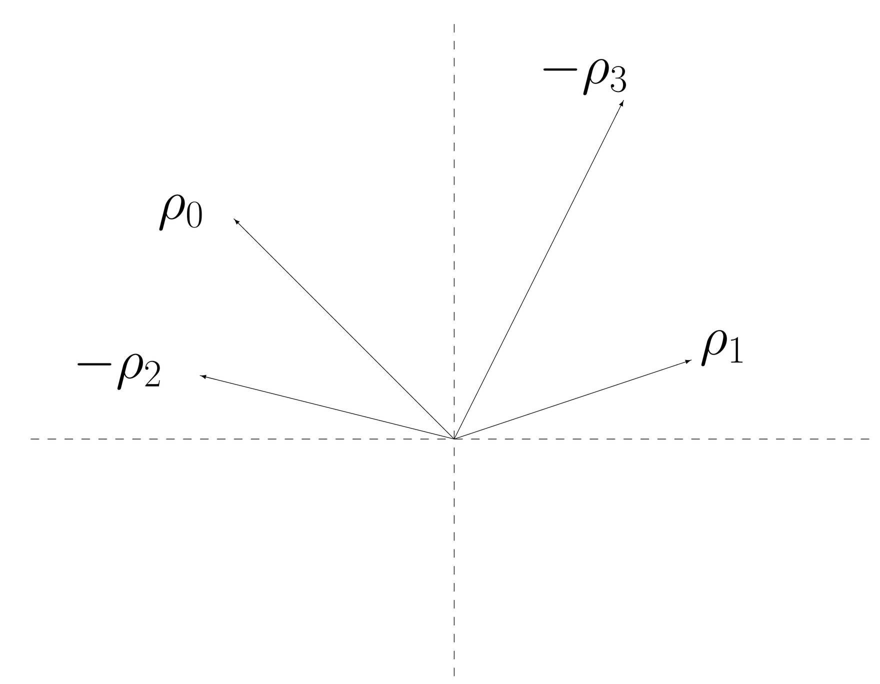

Figure: Configuration of ρ_i for PT-stability conditions

Main Theorem

Theorem. Under the numerical condition $ch_0 \neq 0$ and ch_0, ch_1 coprime, we have:

- the PT-stable objects form a proper Artin stack of finite type, using Lieblich's Artin stack of universally gluable complexes.
- the PT-stable objects form a proper algebraic space of finite type, using Inaba's algebraic space of simple complexes.

What is semistable reduction?

Suppose we start with a flat family I^0 of objects on X over Spec R. If the generic fiber $I^0 \otimes_R K$ is semistable, but the central fiber $I^0 \otimes_R R/\pi$ is not, we can perform an elementary modification on I^0, to obtain a family I^1 whose generic fiber is the same, but whose central fiber is less unstable.

Continuing this process, we obtain an infinite sequence I^i of flat families of objects over Spec R, all of those generic fibers are isomorphic, but with less and less unstable central fibers:

$$\cdots \to I^{i+1} \to I^i \to \cdots \to I^1 \to I^0.$$

If the central fibers of the I^i are all unstable, we can "glue together" the cokernels of the maps $I^i \to I^0$, to obtain a destabilising quotient for the original family I^0, yielding a contradiction. Therefore, we will always arrive at a semistable central fiber (that is, we truly get an R-flat family of semistable objects) after a finite number of modifications. This proves the valuative criterion for completeness.

Proper moduli spaces

Here, we give an example where the valuative criterion for properness holds for PT-stable objects on a three-fold. Properness for Bridgeland-stable objects was previously shown for surfaces by Arcara-Bertram-Lieblich, and for objects in a Noetherian heart by Abramovich-Polishchuk.

Examples of PT-stable objects

Two classes of PT-stable objects on a three-fold X:

- $E = [O_X \xrightarrow{s} F]$ where F is a pure 1-dimensional coherent sheaf, and s has 0-dimensional cokernel. These are the stable pairs in the sense of Pandharipande and Thomas.
- $E = F[1]$ where F is a μ-stable coherent sheaf of homological dimension at most 1.

Further information

A copy of my thesis is available at:
http:// math.stanford.edu/∼jasonlo/

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