Initial Boundary-Value Problems for a Pair of Conservation Laws

By Darryl H. Yong and J. Kevorkian

We describe a multiple-scale technique for solving the initial boundary-value problem over the positive x-axis for a one-dimensional pair of hyperbolic conservation laws. This technique involves decomposing the solution into waves and incorporating slow temporal and stretched spatial scales in different parts of the solution domain. We apply these ideas to a wavemaker problem for shallow water flow and show why the presence of source terms in the conservation laws makes the analytic solution more complicated.

1. Introduction

In this article, we show how to solve the initial boundary-value problem over the positive x-axis for a one-dimensional pair of hyperbolic conservation laws using multiple-scale techniques. These solution techniques apply to weakly nonlinear problems, which commonly arise by perturbing a nonlinear system of equations about a constant steady-state solution.

The novelty of this article is the treatment of problems with both boundary and initial conditions. Kevorkian showed that the appropriate solution strategy for an initial-value problem of this type is to use slow temporal scales (for example, $\tilde{t} = \epsilon t$, where $0 < \epsilon \ll 1$) in addition to the usual x and t scales, to capture any nonlinear effects (Section 8.3.2 of [1]). Signaling problems, in

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which boundary conditions represent signals that propagate into an initially quiescent medium, necessitate the use of stretched spatial scales, such as $\tilde{x} = \epsilon x$ (Section 6.2.4 of [2]). For a problem with both initial and boundary conditions, we anticipate that the correct solution procedure involves both slow temporal and stretched spatial scales. In particular, we show that different scales must be used in different parts of the solution domain. The part of the solution domain influenced solely by the initial conditions requires the use of $\tilde{t}$ only, whereas the part of the solution domain influenced by the boundary condition requires both $\tilde{t}$ and $\tilde{x}$.

In [3], Chikwendu and Easwaran show that this particular combination of temporal and spatial scales is appropriate for weakly nonlinear wave equations of the form

$$u_{tt} - u_{xx} + \epsilon h(u, u_t, u_x) = 0,$$

where $0 < \epsilon \ll 1$. Our work generalizes these findings to encompass pairs of hyperbolic conservation laws that arise naturally in many physical examples. Although it is possible to transform a pair of weakly nonlinear hyperbolic equations to a single weakly nonlinear wave equation for one of the dependent variables, it is more efficient to solve the problem in its original setting.

We begin by studying the one-dimensional wavemaker problem for shallow water flow in Section 2. Next, we generalize our results to a pair of nonlinear, hyperbolic, spatially and temporally homogeneous conservation laws, extending Section 6.2.1 of [2] to include an $\tilde{x}$ dependence. To avoid the possibility of resonant interactions between the dependent variables for certain periodic initial conditions, we do not consider systems of three or more conservation laws [4]. The main lesson from Section 3 is that source terms in the conservation laws produce a perturbation problem that is more difficult to solve analytically. Finally, in Section 4, we discuss the implications of including an $\tilde{x}$ dependence.

2. Wavemaker problem for shallow water flow

To illustrate the main ideas, let us consider a wavemaker problem for one-dimensional shallow-water waves. Let $h(x, t; \epsilon)$ and $u(x, t; \epsilon)$ be the height and velocity of water in a one-dimensional tank. The governing equations,

$$h_t + (uh)_x = 0, \tag{1a}$$

$$u_t + h_x + uu_x = 0, \tag{1b}$$

are derived from physical principles in Section 3.2 of [1]. Because we are more interested in how the solution procedure is affected by boundaries rather than obtaining highly accurate solutions, we have ignored higher-order correction terms in (1b) that account for motion in the vertical direction [2].

Equations (1) have been normalized so that the resting state of the water corresponds to $u = 0$ and $h = 1$. Suppose that our tank is semi-infinitely long with a wavemaker situated near the origin at $x_p = \epsilon p(t)$, where $0 < \epsilon \ll 1$ is our usual small parameter. The wavemaker introduces the boundary condition

$$u(\epsilon p(t), t; \epsilon) = \epsilon p'(t). \tag{2}$$

In addition, let us prescribe some initial height and velocity perturbations:

$$h(x, 0; \epsilon) = 1 + \epsilon g(x), \tag{3a}$$

$$u(x, 0; \epsilon) = \epsilon v(x). \tag{3b}$$

Figure 1 depicts the setup of our one-dimensional tank.

Through Equation (2), we implicitly assume that the wavemaker does not move very much, allowing us to use a Taylor series expansion to replace a moving boundary problem with a fixed boundary problem:

$$\begin{aligned} u(\epsilon p(t), t; \epsilon) &= u(0, t; \epsilon) + \epsilon u_x(0, t; \epsilon)p(t) + \frac{1}{2}\epsilon^2 u_{xx}(0, t; \epsilon)p^2(t) + \mathcal{O}\left(\epsilon^3\right) \\ &= \epsilon p'(t), \end{aligned} \tag{4}$$

as $\epsilon \to 0$. The solution domain for our problem is now the quarter space $x > 0$ and $t > 0$.

We assume that the unknown functions have the asymptotic expansions

$$u(x, t; \epsilon) = \epsilon u^{(1)}(x, \tilde{x}, t, \tilde{t}) + \epsilon^2 u^{(2)}(x, \tilde{x}, t, \tilde{t}) + \mathcal{O}\left(\epsilon^3\right), \tag{5a}$$

$$h(x, t; \epsilon) = 1 + \epsilon h^{(1)}(x, \tilde{x}, t, \tilde{t}) + \epsilon^2 h^{(2)}(x, \tilde{x}, t, \tilde{t}) + \mathcal{O}\left(\epsilon^3\right), \tag{5b}$$

as $\epsilon \to 0$. (Throughout rest of this article, we omit the reminder "$\epsilon \to 0$," which the reader should implicitly assume anytime the symbol $\mathcal{O}(\epsilon^n)$ appears in an

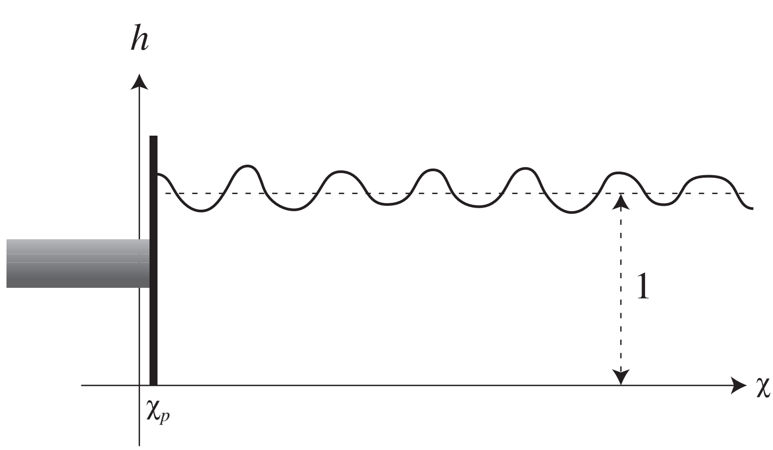

Figure 1. Wavemaker problem for one-dimensional shallow-water waves.

asymptotic expansion.) We include the slow scales $\tilde{x} = \epsilon x$ and $\tilde{t} = \epsilon t$ to capture the nonlinear behavior of (1), and to avoid secular terms that cause our solution to grow linearly in time and space.

2.1. $\mathscr{O}(\epsilon)$ System

Now we substitute (5) into (1) and collect all terms proportional to ϵ, obtaining the $\mathscr{O}(\epsilon)$ system of equations,

$$h_t^{(1)} + u_x^{(1)} = 0, \tag{6a}$$

$$u_t^{(1)} + h_x^{(1)} = 0, \tag{6b}$$

which are subject to the initial and boundary conditions

$$h^{(1)}(x, \tilde{x}, 0, 0) = g(x), \tag{7a}$$

$$u^{(1)}(x, \tilde{x}, 0, 0) = v(x), \tag{7b}$$

and

$$u^{(1)}(0, 0, t, \tilde{t}) = p'(t). \tag{7c}$$

We solve (6) by introducing the characteristic independent variables, $\xi = x - t$ and $\eta = x + t$, along with the characteristic dependent variables, $R^{(i)} = h^{(i)} + u^{(i)}$ and $L^{(i)} = h^{(i)} - u^{(i)}$. We find that

$$R_\eta^{(1)} = L_\xi^{(1)} = 0,$$

which implies that $R^{(1)}$ represents waves traveling to the "right" in the x–t plane, and $L^{(1)}$ represents waves traveling to the "left." We cannot determine anything about the $\tilde{x}$- and $\tilde{t}$-dependencies of $R^{(1)}$ and $L^{(1)}$ until we consider the equations arising at the next order of ϵ. The initial and boundary conditions (7) will be used later.

2.2. $\mathscr{O}(\epsilon^2)$ System

Collecting terms proportional to ϵ^2 yields

$$h_{\tilde{t}}^{(1)} + h_t^{(2)} + u_{\tilde{x}}^{(1)} + \left[u^{(1)}h^{(1)} + u^{(2)}\right]_x = 0,$$

$$u_{\tilde{t}}^{(1)} + u_t^{(2)} + h_{\tilde{x}}^{(1)} + h_x^{(2)} + u^{(1)}u_x^{(1)} = 0,$$

which can be written in terms of characteristic independent and dependent variables as

$$R^{(2)}_{\eta} = -\frac{1}{2}\left[R^{(1)}_{\tilde{t}} + R^{(1)}_{\tilde{x}} + \frac{3}{4}R^{(1)}R^{(1)}_{\xi}\right] + \frac{1}{8}\left(L^{(1)}L^{(1)}_{\eta} + R^{(1)}L^{(1)}_{\eta} + L^{(1)}R^{(1)}_{\xi}\right) \quad (8a)$$

$$L^{(2)}_{\xi} = \frac{1}{2}\left[L^{(1)}_{\tilde{t}} - L^{(1)}_{\tilde{x}} - \frac{3}{4}L^{(1)}L^{(1)}_{\eta}\right] + \frac{1}{8}\left(R^{(1)}R^{(1)}_{\xi} + R^{(1)}L^{(1)}_{\eta} + L^{(1)}R^{(1)}_{\xi}\right). \quad (8b)$$

The quantity in the square brackets in (8a) is independent of η, so simply integrating (8a) with respect to η will result in secular terms. Likewise, the quantity inside the square brackets in (8b) must be set to zero. Therefore, to avoid all secular terms the pair of consistency conditions (sometimes also known as solvability conditions)

$$R^{(1)}_{\tilde{t}} + R^{(1)}_{\tilde{x}} + \frac{3}{4}R^{(1)}R^{(1)}_{\xi} = 0, \quad (9a)$$

and

$$L^{(1)}_{\tilde{t}} - L^{(1)}_{\tilde{x}} - \frac{3}{4}L^{(1)}L^{(1)}_{\eta} = 0, \quad (9b)$$

must be satisfied, subject to the conditions

$$L^{(1)}|_{t=0} = g(x) - v(x), \quad (10a)$$

$$R^{(1)}|_{t=0} = g(x) + v(x), \quad (10b)$$

and

$$R^{(1)}|_{x=0} - L^{(1)}|_{x=0} = 2p'(t). \quad (10c)$$

Note that the boundary condition (10c) now involves a linear combination of $L^{(1)}$ and $R^{(1)}$.

The key to solving (9) is choosing the correct scales for $R^{(1)}$ and $L^{(1)}$. Because $L^{(1)}$ represents waves that are traveling to the left in the x–t plane (toward the wavemaker), these left-going waves are primarily defined by the initial condition (10a) and do not interact with the boundary condition until they meet the wavemaker. Therefore, we should choose scales that are appropriate for an initial-value problem; in other words, we let $L^{(1)} = L^{(1)}(\eta, \tilde{t})$.

The situation for $R^{(1)}$ is a little more complicated, because there are some outgoing waves that are influenced solely by the initial height and velocity perturbation, and there are some that are caused by the wavemaker. To make this distinction clear, we separate the solution domain, $x > 0$ and $t > 0$, into two regions by introducing a positive, monotone increasing function $J(t)$ with $J(0) = 0$ so that $x = J(t)$ is the interface between the two regions. We choose

Region A to have $x = 0$ and $x = J(t)$ as its boundaries, Region B to have $t = 0$ as one of its boundaries (see Figure 2).

We denote $R^{(A)}$ and $R^{(B)}$ for $R^{(1)}$ in Regions A and B, respectively. Because Region B is ahead of the interface, the water there is not yet influenced by the boundary. Therefore, we should choose the scales that are appropriate for an initial-value problem in Region B; in other words, we let $R^{(B)} = R^{(B)}(\xi, \tilde{t})$. In Region A, we will allow $R^{(A)}$ to depend on both $\tilde{x} = \epsilon x$ and $\tilde{t}$ by defining $R^{(A)} = R^{(A)}(\xi, \tilde{x}, \tilde{t})$. To summarize,

$$L^{(1)} = L^{(1)}(\eta, \tilde{t}),$$

$$R^{(1)} = \begin{cases} R^{(A)}(\xi, \tilde{x}, \tilde{t}) & \text{if} \quad x < J(t) \\ R^{(B)}(\xi, \tilde{t}) & \text{if} \quad x > J(t). \end{cases}$$

The governing equations for $L^{(1)}$ and $R^{(B)}$ are

$$R^{(B)}_{\tilde{t}} + \frac{3}{4} R^{(B)} R^{(B)}_{\xi} = 0,$$

and

$$L^{(1)}_{\tilde{t}} - \frac{3}{4} L^{(1)} L^{(1)}_{\eta} = 0,$$

because they do not depend on $\tilde{x}$. These first-order, quasilinear partial differential equations are easily solved using the method of characteristics. Keeping in mind that the initial conditions are given in (10a) and (10b), their solutions are

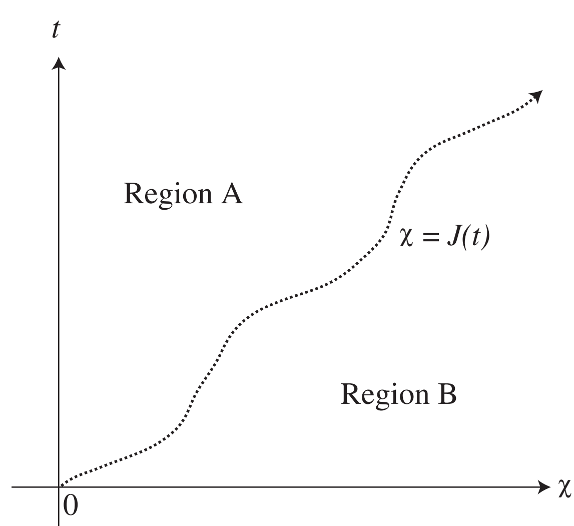

Figure 2. Solution domain divided into two regions.

$$R^{(B)}(\xi, \tilde{t}) = g(\bar{x}) + v(\bar{x}), \quad \text{where} \quad \bar{x} \text{ solves } \xi = \frac{3}{4}\tilde{t}\,[g(\bar{x}) + v(\bar{x})] + \bar{x}, \tag{11a}$$

$$L^{(1)}(\xi, \tilde{t}) = g(\bar{x}) + v(\bar{x}), \quad \text{where } \bar{x} \text{ solves } \eta = -\frac{3}{4}\tilde{t}\,[g(\bar{x}) + v(\bar{x})] + \bar{x}. \tag{11b}$$

Without knowing more about the specific functions $g(x)$ and $v(x)$, these solutions can only be expressed as implicitly defined functions.

Once $L^{(1)}$ is known, the boundary condition (10c) becomes

$$R^{(A)}(-t, 0, \tilde{t}) = L^{(1)}(t, \tilde{t}) + 2p'(t).$$

Using the method of characteristics again, we can now obtain an implicitly defined solution for $R^{(A)}$:

$$R^{(A)}(\xi, \tilde{x}, \tilde{t}) = f\left(-\bar{\xi}, \tilde{t} - \tilde{x}\right), \quad \text{where} \quad \bar{\xi} \text{ solves } \xi = \frac{3}{4}\tilde{x} f\left(-\bar{\xi}, \tilde{t} - \tilde{x}\right) + \bar{\xi}, \tag{12}$$

and $f(t, \tilde{t}) = L^{(1)}(t, \tilde{t}) + 2p'(t)$. At this point, the need for $\tilde{x}$ is evident, because if $R^{(A)}$ is not allowed to depend on $\tilde{x}$, there will be insufficient degrees of freedom to satisfy the initial and boundary conditions in (10).

Because (9a) and (9b) are first-order, quasilinear partial differential equations (sometimes referred to as inviscid Burgers's equations), they admit solutions with shocks. Even when the initial conditions are continuous, wave steepening can lead to shocks forming at later times. When a shock forms, we must return to the integral formulation of the original conservation laws instead of using (9) to determine the shock trajectory.

The fundamental conservation laws for shallow water flow exhibiting the proper flux and conserved quantities are

$$h_t + (uh)_x = 0, \quad \text{(mass)} \tag{13a}$$

$$(uh)_t + \left(u^2 h + \frac{h^2}{2}\right)_x = 0, \quad \text{(momentum)}. \tag{13b}$$

Therefore, the correct shock speed is governed by the pair of equations

$$\frac{ds}{dt}[h]_-^+ = [uh]_-^+, \tag{14a}$$

and

$$\frac{ds}{dt}[uh]_-^+ = \left[u^2 h + \frac{h^2}{2}\right]_-^+, \tag{14b}$$

where $s(t)$ is the shock trajectory and the notation $[\cdot]_-^+$ denotes the value of the jump of a quantity across its discontinuity (see Section 5.3.4 of [1]). For example, if the shock occurs in $R^{(1)}$, substituting the expansion (5) into (14) gives the ordinary differential equation

$$\frac{dK}{d\tilde{\eta}} = \frac{\left[\frac{3}{8}\left(R^{(1)}\right)^2\right]_-^+}{[R^{(1)}]_-^+} + \mathcal{O}(\epsilon) = \frac{3}{8}\left[R^{(A)} + R^{(B)}\right]_{\xi=K(\tilde{\eta})} + \mathcal{O}(\epsilon), \qquad (15)$$

which governs the shock trajectory, here written as $\xi = x - t = K(\tilde{\eta})$, where $\tilde{\eta} = \epsilon\eta$. See Section 6.2.4 of [2] for more details of this derivation.

Now, the only remaining task is to find $J(t)$, the trajectory of the interface between Regions A and B. When the initial velocity perturbation (3b) exactly matches the water velocity imposed by the wavemaker in (2), the interface $J(t)$ is simply the characteristic emanating from the origin, which we know from the form of the characteristic independent variables to be $x = t$. In more realistic situations, the initial condition and boundary condition will not match exactly, and a shock or a fan will result.

The case of a fan is exemplified in a dam-breaking problem in Section 4.3.4 of [1]. In this situation, the solution domain should be divided into three regions: Region A (water under the influence of the wavemaker), Region B (water under the influence of the initial height and velocity perturbations), and a fan region between Regions A and B.

In the case of a shock, the trajectory of the interface between Regions A and B is the shock itself. To determine $x = J(t)$, we solve the differential equation (15) subject to the initial condition $K(0) = 0$ and rewrite the resulting equation in physical variables.

2.3. Numerical verification

To check the validity of our solution technique, we compared our asymptotic solutions with numeric solutions calculated using the CLAWPACK software package [5] written by LeVeque. For our test problem, we chose the initial and boundary conditions

$$u(x, 0; \epsilon) = 0,$$

$$h(x, 0; \epsilon) = 1 + \epsilon\frac{x}{x+1},$$

and

$$u[\epsilon p(t), t; \epsilon] = \epsilon p'(t) \qquad \text{with} \qquad p(t) = \frac{t}{t+1},$$

so that a shock forms because of a mismatch of the velocity at $x = t = 0$ and no additional shocks will form at later times. To simulate a semi-infinite domain, we made the computational domain large enough so that the boundary conditions at the right edge do not affect the part of the solution that is of interest.

Figure 3 shows the numeric and asymptotic solutions five units of time after the initial height and velocity perturbations. We chose $\epsilon = 0.1$, a spatial step size of 0.0004. The agreement between the two solutions is very good except in the vicinity of the shock, because the numeric and asymptotic solutions predict different locations for the shock. The asymptotic solution predicts that the shock lies at $x \approx 5.08$ when $t = 5$, whereas the numerical solution puts the shock near $x \approx 5.26$. This discrepancy is attributable to the fact that the asymptotic solution we have calculated only predicts the shock location to $\mathscr{O}(\epsilon^2 t)$ [see Equation (15), keeping in mind that $d\tilde{\eta}/d\eta = \epsilon$].

Figure 4 demonstrates that the analytic solution is asymptotically correct—it approaches the numerical solution, at the correct rate, as $\epsilon \to 0$. To produce this graph, we compared the asymptotic and numeric solutions for various ϵ at $t = 5$, and measured the error between them over different parts of the solution domain. We use the 1-norm of their difference to calculate this error:

$$\text{absolute error in } h = \int_a^b \left| h_{\text{numeric}}(x,t) - h_{\text{asymptotic}}(x,t) \right| dx.$$

Figure 4a shows the three different regions of the solution domain from which we calculated the absolute errors: regions corresponding to the water behind the

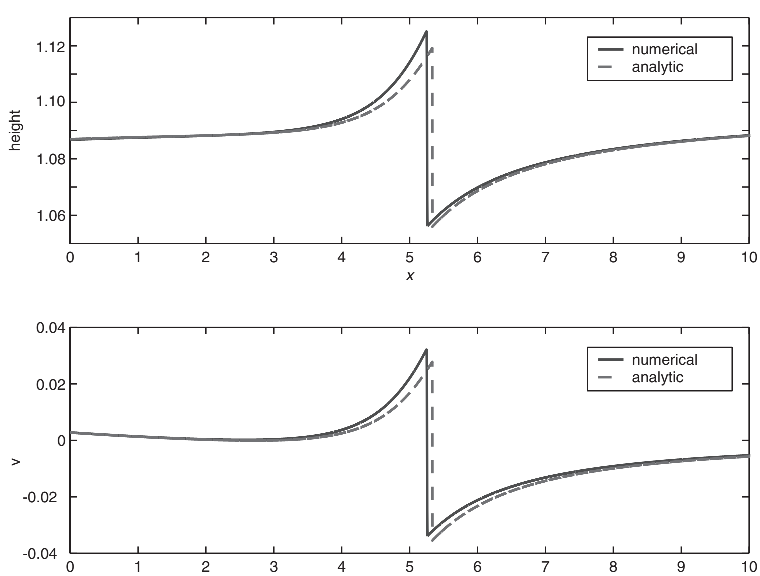

Figure 3. Comparison of a numeric versus asymptotic solution of a wavemaker problem for the shallow-water wave equations at $t = 5$.

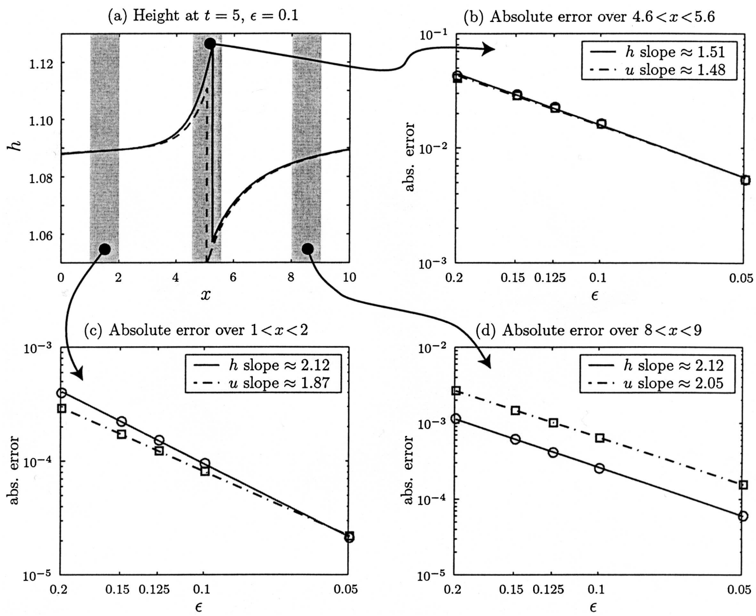

Figure 4. Analysis of error between numeric and calculated solutions over three spatial domains at $t = 5$.

shock (Fig. 4c), on top of the shock (Fig. 4b), and ahead of the shock (Fig. 4d). Note that the absolute error approaches zero like ϵ^2 in the regions away from the shock, which is appropriate, because we have only calculated the $\mathcal{O}(\epsilon)$ contribution to the asymptotic solution. The convergence in the vicinity of the shock is slower, because of the discrepancy of the shock location and the fact that h and u are discontinuous across the shock.

3. General pair of hyperbolic conservation laws

Now that we have seen how to solve initial boundary-value problems for the shallow-water equations, let us apply these solution techniques to a broader class of problems. Consider now a pair of hyperbolic conservation laws written in differential form,

$$\mathbf{p}_t + \mathbf{q}_x = \mathbf{s}, \tag{16}$$

where the conserved quantity $\mathbf{p}$, the flux $\mathbf{q}$, and the source $\mathbf{s}$ are all two-component vectors. Again, let ϵ be a small, positive parameter: $0 < \epsilon \ll 1$.We assume that p_i, q_i, and s_i are functions of the two dependent variables, u_1 and u_2, so that the conservation laws are spatially and temporally homogeneous.

Without source terms, Equation (16) can represent shallow water flow and nearly isentropic gas dynamics (see Section 3.3.4 of [1]). Examples with source terms include glacier flow, chemical exchange processes, chromatography, sedimentation in glaciers, and flow in a channel (see Chapter 3 of [6]).

To obtain a differential equation in terms of the dependent variable, we evaluate derivatives with respect to t and x in (16) to obtain

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = \mathbf{r}(\mathbf{u}), \tag{17}$$

where

$$A(\mathbf{u}) = P^{-1}Q \qquad \text{and} \qquad \mathbf{r}(\mathbf{u}) = P^{-1}\mathbf{s}$$

and P and Q are the Jacobian matrices

$$P = \frac{\partial(p_1, p_2)}{\partial(u_1, u_2)} \qquad \text{and} \qquad Q = \frac{\partial(q_1, q_2)}{\partial(u_1, u_2)}.$$

To make progress on an analytic solution to (17), we assume the existence of a constant steady-state solution, $\mathbf{u}^{(0)}$, about which we construct a perturbation expansion:

$$\mathbf{u}(x, t; \epsilon) = \mathbf{u}^{(0)} + \epsilon\mathbf{u}^{(1)}(x, \tilde{x}, t, \tilde{t}) + \mathbf{u}^{(2)}(x, \tilde{x}, t, \tilde{t}) + \mathcal{O}(\epsilon^3). \tag{18}$$

If the conservation law (16) has no sources, then any pair of constants can serve as a constant steady-state solution; otherwise, the constant steady-state solution must satisfy $\mathbf{s}(\mathbf{u}^{(0)}) = \mathbf{0}$. If the steady-state solution depends on x, or if the original conservation law is not spatially and temporally homogeneous, we end up with a system of linear first-order partial differential equations with coefficients that depend on x or t. At present, there are no analytic methods to solve such problems. However, if we assume that the spatial dependence of the coefficients in (16) is on the fast spatial scale $x^* = x/\epsilon$, then we can make some progress through the theory of multiple-scale homogenization. This is the focus of [7].

With our choice of scales in (18), derivatives with respect to x and t become

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial \tilde{x}} \tag{19a}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial \tilde{t}}. \tag{19b}$$

We expand the matrix A $(\mathbf{u})$ as

$$A(\mathbf{u}) = A\left(\mathbf{u}^{(0)} + \epsilon\mathbf{u}^{(1)} + \ldots\right) = A^{(0)} + \epsilon A^{(1)} + \mathcal{O}\left(\epsilon^2\right) \tag{20}$$

where

$$A^{(0)} = A\left(\mathbf{u}^{(0)}\right)$$

$$A_{ij}^{(1)} = \frac{\partial A_{ij}}{\partial u_m}\left(\mathbf{u}^{(0)}\right)u_m^{(1)}.$$

Similarly, we expand $\mathbf{r}(\mathbf{u})$ as

$$\mathbf{r}(\mathbf{u}) = \mathbf{r}\left(\mathbf{u}^{(0)} + \epsilon\mathbf{u}^{(1)} + \ldots\right) = \mathbf{r}^{(0)} + \epsilon\mathbf{r}^{(1)} + \mathcal{O}\left(\epsilon^2\right) \tag{21}$$

where

$$\mathbf{r}^{(0)} = \mathbf{r}\left(\mathbf{u}^{(0)}\right)$$

$$r_i^{(1)} = \frac{\partial r_i}{\partial u_m}\left(\mathbf{u}^{(0)}\right)u_m^{(1)}.$$

(We adopt the summation convention for repeated indices on m and n.)

We substitute expansions (18), (21), and (20) into (17), use the change of derivatives (19), and collect like powers of ϵ to obtain a hierarchy of equations governing $\mathbf{u}^{(i)}(x, t)$. The governing equations for $\mathbf{u}^{(1)}$,

$$\mathbf{u}_t^{(1)} + A^{(0)}\mathbf{u}_x^{(1)} + B\mathbf{u}^{(1)} = \mathbf{0}, \tag{22}$$

are the most important, because the differential operator represented by the left-hand side determines the behavior of all higher-order corrections to the solution. The entries of the matrix B are given by

$$B_{ij} = \frac{\partial r_i}{\partial u_j}.$$

Using characteristic dependent and independent variables, (22) transforms to

$$\frac{\partial U_1}{\partial \xi_2} + C_{11}U_1 + C_{12}U_2 = 0, \tag{23a}$$

and

$$\frac{\partial U_2}{\partial \xi_1} + C_{21}U_1 + C_{22}U_2 = 0, \tag{23b}$$

where $\xi_i = x - \lambda_i t$, λ_i are the eigenvalues of $A^{(0)}$, and C is similar to B under the same similarity transformation that diagonalizes $A^{(0)}$.

Now we see that effect of source terms in (16) is to couple the pair of equations (23). Although we could make some progress on the analytic solution, it would involve integrals of Bessel functions (see Section 3.7.2 of [1]).

For example, consider again the conservation laws for shallow water flow from the previous section. We can add source terms to (13) by placing the one-dimensional tank on a slight decline away from the wavemaker. The conservation laws become

$$h_t + (uh)_x = 0, \tag{24a}$$

$$(uh)_t + \left(u^2 h + \tfrac{1}{2}h^2\right)_x = h - u^2/F^2, \tag{24b}$$

where F is the Froude constant, which increases with the angle of decline of the channel (see Section 4.3.4 of [1]). The new steady-state solution for this system of equations is $h = 1$, $u = F$, which corresponds to a perfect balance between frictional and gravitational forces. Linearizing about this steady state,

$$h(x, t; \epsilon) = 1 + \epsilon h^{(1)}(x, t) + \mathcal{O}\left(\epsilon^2\right),$$

$$u(x, t; \epsilon) = F + \epsilon u^{(1)}(x, t) + \mathcal{O}\left(\epsilon^2\right),$$

leads to the system of equations

$$\begin{bmatrix} h^{(1)} \\ u^{(1)} \end{bmatrix}_t + \begin{bmatrix} F & 1 \\ 1 & F \end{bmatrix} \begin{bmatrix} h^{(1)} \\ u^{(1)} \end{bmatrix}_x + \begin{bmatrix} 0 & 0 \\ -1 & 2/F \end{bmatrix} \begin{bmatrix} h^{(1)} \\ u^{(1)} \end{bmatrix} = 0,$$

corresponding to (22). As discussed in [8], when $F \approx 2$, this problem provides an example in which the system of equations (23) can be solved sequentially, because $C_{21} = 0$.

The propagation of plane-polarized electromagnetic waves provides us with an example of an essentially coupled hyperbolic system. Instead of deriving a weakly nonlinear problem by perturbing a pair of conservation laws, this instance of (22) comes directly from Maxwell's equations. Let $\mathbf{E} = E(x, t)\mathbf{j}$ and $\mathbf{H} = H(x, t)\mathbf{k}$ be the electric and magnetic fields, respectively. If the current density is exactly proportional to the electric field, $\mathbf{J} = \sigma\mathbf{E}$, where σ is the conductivity of the medium, then E and H satisfy

$$\frac{\partial E}{\partial t} + \frac{1}{\epsilon}\frac{\partial H}{\partial x} + \sigma E = 0, \tag{25a}$$

$$\frac{\partial H}{\partial t} + \frac{1}{\mu}\frac{\partial E}{\partial x} = 0, \tag{25b}$$

where ϵ is the dielectric constant (not to be confused with the small parameter), and μ is the permeability. Note that the presence of σ essentially couples the two equations together, because if the equations are written in terms of characteristic dependent and independent variables, we obtain

$$2\mu^{-1/2}\epsilon^{-1/2}\frac{\partial u_1}{\partial \eta} + \frac{\sigma}{2}(u_1 + u_2) = 0,$$

$$-2\mu^{-1/2}\epsilon^{-1/2}\frac{\partial u_2}{\partial \xi} + \frac{\sigma}{2}(u_1 + u_2) = 0,$$

where $E = \mu^{1/2}(u_1 + u_2)$, $H = \epsilon^{1/2}(u_1 - u_2)$, $\xi = x - \mu^{-1/2}\epsilon^{-1/2}t$, and $\eta = x + \mu^{-1/2}\epsilon^{-1/2}t$. Although the general solution to these equations can be written in terms of Bessel functions, obtaining the solvability conditions at the next order of ϵ becomes extremely cumbersome.

If source terms are absent in the original conservation laws, we can easily apply the methods from the last section. Using the eigenvalues and eigenvectors of $A^{(0)}$, we use characteristic dependent and independent variables to decompose the solution into left-going and right-going waves. (This decomposition relies on the fact that $A^{(0)}$ has one positive and one negative eigenvalue, something that can be accomplished through an appropriate change of independent variables.) The solution domain for the right-going wave must be divided into two regions, separating points in the solution domain by their dependence on initial and boundary data. We then include the appropriate temporal and spatial scales for each wave and solve quasilinear first-order partial differential equations to obtain the solution to as high a degree of ϵ as desired. Finally, we determine the curve that divides the two regions of the solution domain.

4. Discussion

All of the problems solved in this article have the feature that the $\mathcal{O}(\epsilon)$ solution can be decomposed into two waves, each traveling in a different direction (for example $L^{(1)}$ and $R^{(1)}$ for the wavemaker problem). The wave that is traveling toward the boundary condition is primarily determined by its initial condition, so it is the wave that travels away from the boundary that has the responsibility of satisfying the boundary condition. In these problems, we have used the stretched spatial scale $\tilde{x}$ to give this "outbound" wave the extra freedom to satisfy the boundary condition.

Let us look more closely at how the extra freedom is achieved. The effect of including $\tilde{x}$ first presents itself in the consistency conditions. For example, in the wavemaker problem, the consistency conditions for the wave traveling to the right are

$$R_{\tilde{t}}^{(A)} + R_{\tilde{x}}^{(A)} + \frac{3}{4} R^{(A)} R_{\xi}^{(A)} = 0, \tag{26a}$$

and

$$R_{\tilde{t}}^{(B)} + \frac{3}{4} R^{(B)} R_{\xi}^{(B)} = 0. \tag{26b}$$

We do not include $\tilde{x}$ in (26b), because the water in Region B is primarily influenced by the initial conditions, and the appropriate solution procedure for an initial value problem is to add the slow temporal scale $\tilde{t}$ only. What is the difference between (26a) and (26b)?

First, we point out that once we convert (26a) and (26b) back to physical coordinates, they actually represent very similar equations. Let $f(\xi, \tilde{x}, \tilde{t}) = R^{(1)} + \epsilon R^{(2)} + \ldots$ in Region A and $g(\xi, \tilde{t}) = R^{(1)} + \epsilon R^{(2)} + \ldots$ in Region B represent the wave traveling to the right, with the contributions from all orders of ϵ combined.

In Equation (26a), the variables $\tilde{t}$, $\tilde{x}$, and ξ are not really independent variables, because $\xi = x - t = (\tilde{x} - \tilde{t})/\epsilon$. Because it is not possible to change from three independent variables to two independent variables, we must maintain the formalism that ξ is independent of $\tilde{t}$ and $\tilde{x}$. The correct change of variables requires us to consider two separate sets of variable changes: first the change from $\tilde{x}$ and $\tilde{t}$ to x and t, then the change from $\xi = x - t$ and $\eta = x + t$ to x and t. When we perform these changes, (26a) becomes

$$F_t + F_x + \frac{\epsilon}{2} F(F_x - F_t) + \mathcal{O}\left(\epsilon^2\right) = 0, \tag{27}$$

where $F(x, t) = f(\xi, \tilde{x}, \tilde{t})$.

Because (26b) only contains two independent variables, the correct change of variables involves the relationships

$$\tilde{t} = \epsilon t \qquad t = \tilde{t}/\epsilon$$

$$\xi = x - t \qquad x = \xi + \tilde{t}/\epsilon.$$

After some algebra, (26b) becomes

$$G_t + G_x + \epsilon G G_x + \mathcal{O}\left(\epsilon^2\right) = 0, \tag{28}$$

where $G(x, t) = G(\xi, \tilde{x}, \tilde{t})$. Although (27) and (28) look different, once we use the fact

$$\frac{1}{2} G_t + \frac{1}{2} G_x + \mathcal{O}(\epsilon) = 0$$

in (28), the two equations match to $\mathcal{O}(\epsilon)$.

So we see that the addition of $\tilde{x}$ does not significantly affect the qualitative behavior of the outgoing wave, because the governing equations with and without $\tilde{x}$ are essentially the same once we revert to physical independent variables. The effect of adding $\tilde{x}$, therefore, can only be seen while maintaining the formalism that ξ and η are independent of $\tilde{x}$ and $\tilde{t}$. For example, when we used the method of characteristics to obtain $R^{(A)}$ in (12), we considered $\tilde{t}$, $\tilde{x}$, and ξ to be three independent variables.

To summarize, the presence of boundaries necessitates the addition of stretched spatial scales to our multiple-scale solution so that there are enough degrees of freedom to satisfy all initial and boundary conditions. These stretched spatial scales do not significantly affect the qualitative behavior of the solution, and their benefit is only achieved by solving consistency conditions under the assumption that all scales are independent of one another.

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References

1. J. Kevorkian, *Partial Differential Equations: Analytical Solution Techniques*, 2nd ed., Springer, New York, 2000.
2. J. Kevorkian and J. D. Cole, *Multiple Scale and Singular Perturbation Methods*, vol. 114 of Applied Mathematical Sciences, 2nd ed., Springer, New York, 1996.
3. S. C. Chikwendu and C. V. Easwaran, Multiple-scale solution of initial-boundary value problems for weakly nonlinear wave equations on the semi-infinite line, *SIAM J. Appl. Math.* 52:958–964 (1992).
4. J. Kevorkian, Weakly nonlinear conservation laws with source terms, in *Mathematics Is for Solving Problems* (L. P. Cook, V. Roytburd, and M. Tulin, Eds.), SIAM, Philadelphia, 1996, 167–178.
5. R. J. LeVeque, *CLAWPACK version 4.0 user guide*. Department of Applied Mathematics, University of Washington, 1999. CLAWPACK software and documentation are available from the web at http://www.amath.washington.edu/~claw.
6. G. B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1974.
7. D. H. Yong and J. Kevorkian, Solving boundary-value problems for systems of hyperbolic conservation laws with rapidly varying coefficients, *Studies Appl. Math*. 108:259–303 (2002).
8. J. Kevorkian, J. Yu, and L. Wang, Weakly nonlinear waves for a class of linearly unstable hyperbolic conservation laws with source terms, *SIAM J. Appl. Math*. 55:446–484 (1995).

Recommended readings

M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*, Dover, New York, 1970.

S. C. Chikwendu and J. Kevorkian, A pertubation method for hyperbolic equations with small nonlinearities, *SIAM J. Appl. Math*. 22:235–258 (1972).

C. V. Easwaran, A scaled characteristics method for the asymptotic solution of weakly nonlinear equations, *Electron. J. Differen. Eqs.* (1998).

T. R. Fogarty, High-resolution finite volume methods for acoustics in a rapidly heterogeneous medium, Master's thesis, University of Washington, 1997.

A. S. Fokas and B. Pelloni, Method of solving moving boundary value problems for linear evolution equations, *Phys. Rev. Lett.* 84:4785–4789 (2000).

J. Kevorkian and D. L Bosley, Multiple-scale homogenization for weakly nonlinear conservation laws with rapid spatial fluctuations, *Studies Appl. Math*. 101:127–183 (1998).

R. J. LeVeque, *Numerical Methods for Conservation Laws*, Birkhauser, Basel, 1992.

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