Solving Boundary-Value Problems for Systems of Hyperbolic Conservation Laws with Rapidly Varying Coefficients

By Darryl H. Yong and J. Kevorkian

We study how boundary conditions affect the multiple-scale analysis of hyperbolic conservation laws with rapid spatial fluctuations. The most significant difficulty occurs when one has insufficient boundary conditions to solve consistency conditions. We show how to overcome this missing boundary condition difficulty for both linear and nonlinear problems through the recovery of boundary information. We introduce two methods for this recovery (multiple-scale analysis with a reduced set of scales, and a combination of Laplace transforms and multiple scales) and show that they are roughly equivalent. We also show that the recovered boundary information is likely to contain secular terms if the initial conditions are nonzero. However, for the linear problem, we demonstrate how to avoid these secular terms to construct a solution that is valid for all time. For nonlinear problems, we argue that physically relevant problems do not exhibit the missing boundary condition difficulty.

1. Introduction

In this article, we study one-dimensional, hyperbolic conservation laws with rapidly fluctuating coefficients over the semi-infinite domain. We are primarily

Address for correspondence: Professor Darryl H. Yong, Department of Applied and Computational Mathematics, MC 217-50, Caltech, Pasadena, CA 91125. E-mail: dyong@its.caltech.edu.

STUDIES IN APPLIED MATHEMATICS 108:259–303

Published by Blackwell Publishing, 350 Main Street, Malden, MA 02148, USA, and 108 Cowley Road, Oxford, OX4 1JF, UK.

interested in the role that boundary conditions play in the multiple-scale analysis of these problems, developed by Kevorkian and Bosley in [1] to study initial-value problems on the infinite domain.

Hyperbolic conservation laws, which typically arise from such physical principles as the conservation of mass, momentum, or energy, govern the propagation of information at finite speeds through some medium. The governing equations for shallow water waves and acoustics are canonical examples.

When properties of the medium being modeled have rapid spatial variations, the governing hyperbolic conservation laws typically exhibit these variations in their coefficients. For example, wave propagation in a bubbly liquid [2, 3], and gravity waves in a channel with a rough bottom [4] are modeled using hyperbolic conservation laws with rapid spatial fluctuations. If these fluctuations are large (for example, the amplitude of fluctuations being comparable to its average value), regular perturbation techniques cannot be used. In such situations, the exact solution to the conservation laws cannot be derived analytically. Instead, we seek to determine the qualitative effects of these rapid spatial fluctuations or to find "effective" equations that describe the homogenized (averaged) behavior of the system.

The literature in the area of homogenization is vast and encompasses many disciplines of mathematics. Systematic treatment of the theory of homogenization began with the work of Bensoussan, Lions, and Papanicolaou in 1978, although the application of multiple scales to the theory of homogenization was anticipated by J. B. Keller [5]. Another standard reference in this field is Bakhvalov and Panasenko [6]. For a list of specific applications of homogenization techniques, see Kevorkian and Bosley [1].

1.1. Missing boundary condition difficulty

Almost all physically relevant problems involve boundaries of some sort and, therefore, necessitate the enforcement of boundary conditions. For a given system of conservation laws, we can specify only a certain number of boundary conditions for a well-posed problem. However, the equations that describe the homogenized behavior of the system are typically of higher order than the original problem, and therefore, require more boundary conditions than can be specified without ill-posedness. In [7], Santosa and Symes encounter this "missing boundary condition difficulty," in the context of using Bloch wave expansions to derive an effective equation for wave propagation in a periodic composite material. They note that "in the presence of boundaries, none of what is discussed up to now is valid, some other approach will be necessary."

1.2. Proposed work

In Section 2, we recount how to write a general system of hyperbolic conservation laws in a standard form. We also discuss how initial conditions

must be carefully chosen to avoid solutions that depend on the fast temporal scale $t^* = t/\epsilon$. We then present two methods for analyzing systems of hyperbolic conservation laws with rapid spatial fluctuations. The first is the usual multiple-scale method (Section 3), and the second is a combination of Laplace transforms and multiple-scale analysis (Section 4). Using both methods, we highlight the missing boundary condition difficulty and show how to overcome it through the recovery of boundary information.

2. Problem description

In this section, we describe a standard form for analyzing systems of conservation laws with rapidly fluctuating quantities and discuss how initial conditions must be carefully chosen to avoid solutions that depend on $t^* = t/\epsilon$.

2.1. Problem set-up

In this article, we consider partial differential equations of the form

$$\mathbf{u}_t + A(x^*)\mathbf{u}_x + B(x^*)\mathbf{u} = \epsilon[C(\mathbf{u}, x^*)\mathbf{u} + D(\mathbf{u}, x^*)\mathbf{u}_x] + \mathcal{O}(\epsilon^2), \qquad (1)$$

where the vector of unknown functions is $\mathbf{u}$, and A, B, C, D are all matrices with C and D depending linearly in $\mathbf{u}$. In [1], Kevorkian and Bosley show that this is the most general form of partial differential equations that arises when we perturb a general, one-dimensional, quasilinear system of conservation laws about a known steady-state solution. Furthermore, the original conservation laws may depend explicitly on the fast spatial variable $x^* = x/\epsilon$. (See [1] for many examples and for the precise definitions of A, B, C, D in terms of the original conservation laws.)

We use the standard form (1) repeatedly throughout this article, because it gives us a unified way of analyzing systems of hyperbolic conservation laws. In particular, we only consider pairs of conservation laws, so all of the matrix coefficients in (1) are 2×2 matrices. We do not consider systems of three or more conservation laws to avoid the possibility of resonant interactions between the dependent variables for certain periodic initial conditions [8].

Furthermore, all of our examples will involve periodic x^*-fluctuations in (1). There is nothing that prevents us from assuming any other type of fluctuations—the only requirement is that the fluctuating functions are well behaved so that the averaging operators defined in Appendix A can be applied to them. Although choosing periodic fluctuations makes the algebra a little simpler, it allows for the possibility of resonant interactions between waves traveling through the medium and the medium itself. This phenomenon was first observed in [1] and discussed in more detail in [9].

Because the nonlinear terms in (1) are premultiplied by ϵ, it is the linear problem

$$\mathbf{u}_t + A(x^*)\mathbf{u}_x + B(x^*)\mathbf{u} = 0 \tag{2}$$

that governs the basic behavior of its solutions. That is why it is important to first look carefully at (2) before studying the general nonlinear problem.

The eigenvalues of $A(x^*)$ are the speeds at which information propagates through the system, so for (2) to describe a hyperbolic system of partial differential equations, the matrix $A(x^*)$ must have real and distinct eigenvalues for all x^*.

Let us also assume that one of the eigenvalues is always positive and one is always negative. The effect of this assumption is to turn a boundary such as $x = 0$ into a space-like arc, a contour in the x-t plane that only has one characteristic emanating from it in positive time. Therefore, a well-posed boundary-value problem can only have one condition on one linear combination of the dependent variables at $x = 0$. In contrast, the solution boundary $t = 0$ is a time-like arc, because it has two characteristics emanating from it in positive time. The reason for making this assumption about the eigenvalues will become clearer when we highlight the missing boundary condition difficulty in Section 3.

2.2. Choosing the initial conditions for a t^*-independent solution

In preparation for the analysis in the next section, we now show how to choose initial conditions for (1) so that the solution is independent of $t^* = t/\epsilon$.

Why is it important to have solutions that are independent of t^*? If the functions represented by *A, B, C*, and *D* in (1) depend on x, the natural spatial scale of the problem, a closed-form solution to (1) would be out of reach. It is the assumption that these functions depend on x^* that allows us to make some progress through multiple-scale analysis. Likewise, if the solution to (1) depends on t^*, then it is possible to use a simple change of variables to rescale the problem so that the natural fluctuations are on the same scale as the fluctuations of the coefficient matrices. Therefore, it is important to ensure that solutions to (1) are independent of the fast temporal scale $t^* = t/\epsilon$.

In [7], Santosa and Symes tacitly assume that their solution is independent of the fast temporal scale, although not in the context of multiple-scale analysis. Their analysis of the linear wave equation

$$\rho(x^*)w_{tt} - [k(x^*)w_x]_x = 0, \tag{3}$$

where $w(x,t;\epsilon)$ is the displacement, $\rho(x^*)$ the density, and $k(x^*)$ the bulk modulus of the medium, involves a Bloch wave expansion—essentially a spectral decomposition of the partial differential operator for (3) that recasts the initial-value problem as an eigenvalue problem. They express their solution as

$$w(x,t;\epsilon) = \int_{|k|\le 2\pi} g(k)\exp[ikx \pm i\omega(k)t]dk + \mathcal{O}(\epsilon),$$

where $\omega(k)$ are the eigenvalues associated with each Bloch wave. The important assumption here is that k, a scaled wavenumber, is restricted by the integration limits. This assumption that the solution is band limited is equivalent to assuming that the solution is independent of the fast temporal scale t^*.

So how do we ensure that the solution is independent of t^*? According to Kevorkian and Bosley, only problems with special initial conditions to higher orders lead to solutions without high-frequency temporal oscillations [1]. Determining the necessary x^*-dependence is easy—from the calculated multiple-scale solution, we set $t = 0$ to see what initial conditions are necessary to support that solution. As expected, the leading-order initial conditions must be independent of x^* for us to make any progress on the solution to (1). The x^*-dependencies of all higher-order terms in the initial conditions cannot be arbitrarily chosen. [For example, refer to Equation (31).] Practically speaking, this means that any solution that we obtain using multiple scales is actually the solution to a nearby problem: the smaller ϵ gets, the better our approximation becomes.

Finally, we also cite the finding that discontinuous initial data lead to solutions that are x^*- and t^*-dependent in the region of influence of the discontinuity [1]. This finding means we should also ensure that our initial and boundary data are continuous functions and that they match up correctly at any places in the solution domain where they meet. Furthermore, problems whose solutions exhibit shocks require the use of x^* and t^* after the time of shock formation.

3. Multiple-scale analysis

In this section, we apply multiple-scale analysis to problems in the standard form (1). We begin by analyzing the linear wave equation to demonstrate the multiple-scale method, the difficulty of the missing boundary conditions, and how to overcome it. The ideas from this linear problem are then applied to more general nonlinear systems of conservation laws.

3.1. The linear wave equation

Before we analyze the general nonlinear problem, we begin by studying the linear wave equation, written as a second-order partial differential equation,

$$\rho(x^*)w_{tt} - [k(x^*)w_x]_x = 0. \tag{4}$$

Here, $w(x,t;\epsilon)$ is the displacement, and we allow the density $\rho(x^*)$ and bulk modulus of the medium $k(x^*)$ to vary rapidly on the $x^* = x/\epsilon$ scale. The description of linear (small amplitude) acoustic waves in a stationary gas is one of many applications of the linear wave equation. In that context, u_1 and u_2

stand for the velocity and pressure, respectively, and the speed of sound is $c(x^*) = \sqrt{k(x^*)/\rho(x^*)}$.

Changing variables using $u_1 = w_t$ and $u_2 = -kw_x$ puts (4) in the form

$$\mathbf{u}_t + A(x^*)\mathbf{u}_x = 0, \tag{5}$$

with

$$A(x^*) = \begin{bmatrix} 0 & 1/\rho(x^*) \\ k(x^*) & 0 \end{bmatrix}. \tag{6}$$

The functions $k(x^*)$ and $\rho(x^*)$ are always positive, so the eigenvalues of $A(x^*)$ are real and distinct, and the problem is hyperbolic.

Although we focus specifically on the linear wave equation throughout Section 3.1, our analysis is general enough to encompass (5) with any $A(x^*)$, which corresponds to the linear part of (1) with $B(x^*) = 0$.

Our solution procedure consists of

1. expanding **u** into an asymptotic series containing all of the spatial and temporal scales that we use throughout the problem;
2. using averaging operators to separate terms that depend on x^* from the terms that do not;
3. removing potentially secular terms from the x^*-homogenized (averaged) equations to produce long-term evolution equations (consistency conditions); and
4. solving the long-term evolution equations, using recovered boundary information, if necessary.

This analysis results in a qualitative understanding of the long-term behavior and explicit expressions for the averaged (homogenized) solution.

This method differs slightly from that outlined by Kevorkian and Bosley [1], in which their asymptotic expansion for **u** only incorporates x^*, x, and t. The resulting homogenized equations tell us the cumulative effects of the fluctuations in $A(x^*)$, but to solve the homogenized equations actually, we must further expand the solution with the desired slow temporal and stretched spatial scales. Here, we incorporate both expansions from the beginning.

Choosing the correct scales to include in our multiple-scale analysis is the key to the whole problem. Because the goal of homogenization theory is to characterize the macroscopic behavior of media with microscopic structure, we anticipate that the cumulative effects of the fluctuations of the media on the $x^* = x/\epsilon$ scale will have long-term effects on the solution. If we want to capture these effects, we must include some combination of slow temporal scales and stretched spatial scales into our multiple-scale analysis.

Precisely how slow or stretched our scales should be is largely determined by the type of behavior we want to capture. The long-term evolution equation

arises from the removal of potentially secular terms, and it tells us how the solution depends on the slow or stretched scales. Therefore, including different slow and stretched scales leads to different long-term evolution equations. In general, using $\epsilon^n x$ or $\epsilon^n t$ enables us to obtain a long-term evolution equation from the $\mathcal{O}(\epsilon^n)$ system of x^*-homogenized equations. In this problem, we want to take into account the cumulative dispersive effects of the fluctuations in $A(x^*)$, so we use the asymptotic expansion

$$\mathbf{u}(x,t;\epsilon) = \mathbf{u}^{(0)}(x^*,x,t,\hat{t}) + \epsilon\mathbf{u}^{(1)}(x^*,x,t,\hat{t}) + \ldots.$$

We do not include any stretched spatial scales, because they are not necessary in a linear problem such as (5). With this choice of scales, derivatives with respect to x and t become

$$\frac{\partial}{\partial x} \to \epsilon^{-1}\frac{\partial}{\partial x^*} + \frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon^2\frac{\partial}{\partial \hat{t}.}$$

(This change of variables is actually an abuse of notation—to eliminate confusion, we should really use x to stand for the original spatial scale, and some other variable such as $y = x$ to stand for the same scale after the other spatial scales are introduced.) We plug

$$\mathbf{u}_t \to \mathbf{u}_t^{(0)} + \epsilon\mathbf{u}_t^{(1)} + \epsilon^2(\mathbf{u}_{\hat{t}}^{(0)} + \mathbf{u}_t^{(2)}) + \ldots$$

and

$$\mathbf{u}_x \to \epsilon^{-1}\mathbf{u}_{x^*}^{(0)} + (\mathbf{u}_x^{(0)} + \mathbf{u}_{x^*}^{(1)}) + \epsilon(\mathbf{u}_x^{(1)} + \mathbf{u}_{x^*}^{(2)}) + \epsilon^2(\mathbf{u}_x^{(2)} + \mathbf{u}_{x^*}^{(3)}) + \ldots$$

into the original Equation (5) and separate terms according to their associated power of ϵ.

3.1.1. *$\mathcal{O}(\epsilon^{-1})$ System.* When we collect all terms that are proportional to ϵ^{-1}, we obtain $A(x^*)\,\mathbf{u}_{x^*}^{(0)} = \mathbf{0}$. Because the eigenvalues of $A(x^*)$ are never zero, $A(x^*)$ may be inverted to obtain $\mathbf{u}_{x^*}^{(0)} = \mathbf{0}$. In other words, the leading-order behavior of the solution does not depend on the fast spatial scale. We follow the convention that all quantities independent of the fast spatial scale are underlined. For example, we denote $\mathbf{u}^{(0)} = \underline{\mathbf{u}}^{(0)}(x,t,\hat{t})$.

3.1.2. *$\mathcal{O}(1)$ System.* The $\mathcal{O}(1)$ system of equations,

$$\mathbf{u}_{x^*}^{(1)} = -A^{-1}\underline{\mathbf{u}}_t^{(0)} - \underline{\mathbf{u}}_x^{(0)}, \tag{7}$$

may be easily integrated with respect to x^*, but before we do this, we need to separate the terms that depend on the fast spatial scale from the terms that do

not. When x^*-independent terms are integrated with respect to x^*, terms proportional to x^* arise. Such terms, examples of so-called secular terms, are not allowed, because they cause the asymptotic expansion to become nonuniform for large x^*. In other words, as x^* increases, the terms in the asymptotic expansion outgrow their assigned orders of magnitude.

To separate the x^*-dependent terms from the x^*-independent terms, we use the averaging operators defined in Appendix A. Setting the average (x^*-independent) part of the right-hand side of (7) to zero, we get the x^*-homogenized equation

$$\underline{\mathbf{u}}_t^{(0)} + \langle A^{-1}\rangle^{-1}\underline{\mathbf{u}}_x^{(0)} = \mathbf{0}. \tag{8}$$

The multiple-scale analysis shows that the correct description of the average behavior of our system is obtained by replacing the fluctuating $A(x^*)$ with the constant matrix $\langle A^{-1}\rangle^{-1}$; the naive approach of replacing $A(x^*)$ with $\langle A(x^*)\rangle$ is incorrect. In the context of the linear wave equation,

$$\langle A^{-1}\rangle^{-1} = \begin{bmatrix} 0 & 1/\langle p(x^*)\rangle \\ \langle 1/k(x^*)\rangle^{-1} & 0 \end{bmatrix}.$$

It is interesting that although the density is replaced by its average value, the bulk modulus must be replaced by its harmonic average to produce the correct averaged behavior.

The remaining (fluctuating) part of the $\mathcal{O}(1)$ system is

$$\mathbf{u}_{x^*}^{(1)} = -\{A^{-1}(x^*)\}\underline{\mathbf{u}}_t^{(0)},$$

which, when integrated, becomes

$$\mathbf{u}^{(1)} = -[\![A^{-1}(x^*)]\!]\underline{\mathbf{u}}^{(0)} + \underline{\mathbf{u}}^{(1)}(x,t,\hat{t}), \tag{9}$$

where $\underline{\mathbf{u}}^{(1)}(x,t,\hat{t})$ is the constant of integration.

3.1.3. $\mathcal{O}(\epsilon)$ *System*. The governing equations for $\mathbf{u}^{(1)}$ are

$$\mathbf{u}_t^{(1)} + A(x^*)\left(\mathbf{u}_x^{(1)} + \mathbf{u}_{x^*}^{(2)}\right) = \mathbf{0}.$$

We plug in (9) and rearrange to get

$$\mathbf{u}_{x^*}^{(2)} = -A^{-1}\underline{\mathbf{u}}_t^{(1)} - \underline{\mathbf{u}}_x^{(1)} + [\![A^{-1}]\!]\underline{\mathbf{u}}_{tx}^{(0)} + A^{-1}[\![A^{-1}]\!]\underline{\mathbf{u}}_{tt}^{(0)}.$$

The removal of secular terms produces the equation

$$\underline{\mathbf{u}}_t^{(1)} + \langle A^{-1}\rangle^{-1}\underline{\mathbf{u}}_x^{(1)} = \langle A^{-1}\rangle^{-1}\langle A^{-1}[\![A^{-1}]\!]\rangle\underline{\mathbf{u}}_{tt}^{(0)}. \tag{10}$$

The remaining (fluctuating) part of the $\mathcal{O}(\epsilon)$ system is

$$\mathbf{u}^{(2)}_{x^*} = -\{A^{-1}\}\underline{\mathbf{u}}^{(1)}_t + [\![A^{-1}]\!]\underline{\mathbf{u}}^{(0)}_{tx} + \{A^{-1}[\![A^{-1}]\!]\}\underline{\mathbf{u}}^{(0)}_{tt},$$

which, when integrated, becomes

$$\mathbf{u}^{(2)} = -[\![A^{-1}]\!]\underline{\mathbf{u}}^{(1)}_t + [\![[\![A^{-1}]\!]]\!]\underline{\mathbf{u}}^{(0)}_{tx} + [\![A^{-1}[\![A^{-1}]\!]]\!]\underline{\mathbf{u}}^{(0)}_{tt} + \underline{\mathbf{u}}^{(2)}(x,t,\hat{t}), \qquad (11)$$

where $\mathbf{u}^{(2)}(x,t,\hat{t})$ is the constant of integration.

3.1.4. $\mathcal{O}(\epsilon^2)$ *System*. For brevity, we only show here the homogenized equation governing $\underline{\mathbf{u}}^{(2)}$,

$$\begin{aligned}\underline{\mathbf{u}}^{(0)}_{\hat{t}} + \underline{\mathbf{u}}^{(2)}_t + \langle A^{-1}\rangle^{-1}\underline{\mathbf{u}}^{(2)}_x = {} & \langle A^{-1}\rangle^{-1}\langle A^{-1}[\![A^{-1}]\!]\rangle\underline{\mathbf{u}}^{(1)}_{tt} \\ & - \langle A^{-1}\rangle^{-1}\langle A^{-1}[\![[\![A^{-1}]\!]]\!]\rangle\underline{\mathbf{u}}^{(0)}_{ttx} \\ & - \langle A^{-1}\rangle^{-1}\langle A^{-1}[\![A^{-1}[\![A^{-1}]\!]]\!]\rangle\underline{\mathbf{u}}^{(0)}_{ttt}, \quad (12)\end{aligned}$$

which is obtained by removing secular terms from the $\mathcal{O}(\epsilon^2)$ system of equations.

This process of removing terms independent of x^* before integrating can be repeated to as high as degree of ϵ as desired. However, we soon see that, with our current choice of scales (in particular, $\hat{t} = \epsilon^2 t$), it is not necessary to proceed any further.

3.1.5. *Solving the homogenized equations for an initial-value problem.* All of the analysis done up to this point is applicable to problems with and without boundary conditions. Before we tackle the difficulties that boundaries add to this problem, we first demonstrate that our solution procedure works for the initial-value problem. We are most concerned about whether we can generate homogenized equations that accurately describe the long-term evolution of Equation (5).

These long-term evolution equations arise while solving the x^*-homogenized equations from each order of ϵ. It is only now that we begin to see how different $A(x^*)$ lead to different types of long-term evolution equations and necessitate different choices of slow temporal or stretched spatial scales.

To solve the x^*-homogenized equation governing $\underline{\mathbf{u}}^{(0)}$, (8), we diagonalize $\langle A^{-1}\rangle^{-1} = P\Lambda P^{-1}$ so that

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Without loss of generality, we choose λ_1 to be the positive eigenvalue, and λ_2 to be the negative one.

Making the substitution $\underline{\mathbf{u}}^{(i)} = P\underline{\mathbf{w}}^{(i)}$ (the $\underline{\mathbf{w}}^{(i)}$ variables are called the characteristic dependent variables) and multiplying on the left by P^{-1}, we get

$$\underline{\mathbf{w}}_t^{(0)} + \Lambda \underline{\mathbf{w}}_x^{(0)} = \mathbf{0},$$

which can be written out as

$$\underline{w}_{1t}^{(0)} + \lambda_1 \underline{w}_{1x}^{(0)} = 0,$$

$$\underline{w}_{2t}^{(0)} + \lambda_2 \underline{w}_{2x}^{(0)} = 0.$$

These equations imply that $\underline{w}_1{}^{(0)} = \underline{w}_1{}^{(0)}(\xi, t)$, and $\underline{w}_2{}^{(0)} = \underline{w}_2{}^{(0)}(\eta, t)$, where $\xi = x - \lambda_1 t$ and $\eta = x - \lambda_2 t$ are called the characteristic independent variables. The dependence of $\underline{\mathbf{w}}^{(0)}$ on the slow time scale $\hat{t}$ cannot be determined until we consider the equations arising at higher orders.

We rewrite the $\mathcal{O}(\epsilon)$ system of x*-homogenized equations (10) in terms of the characteristic dependent variables to get

$$\underline{\mathbf{w}}_t^{(1)} + \Lambda \underline{\mathbf{w}}_x^{(1)} = \mathcal{M} \underline{\mathbf{w}}_{tt}^{(0)},$$

where $\mathcal{M} = P^{-1}\langle A^{-1}\rangle^{-1}\langle A^{-1}[\![A^{-1}]\!]\rangle P$. Recalling that $\underline{w}_1^{(0)}$ does not depend on η and $\underline{w}_2^{(0)}$ does not depend on ξ, and using the facts $\partial_x \rightarrow \partial_\xi + \partial_\eta$ and $\partial_t \rightarrow -\lambda_1 \partial_\xi - \lambda_2 \partial_\eta$, we can rewrite the equation above using characteristic independent variables:

$$\begin{bmatrix} (\lambda_1 - \lambda_2) w_{1\eta}^{(1)} \\ (\lambda_2 - \lambda_1) w_{2\xi}^{(1)} \end{bmatrix} = \mathcal{M} \begin{bmatrix} \lambda_1^2 w_{1\xi\xi}^{(0)} \\ \lambda_2^2 w_{2\eta\eta}^{(0)} \end{bmatrix}. \tag{13}$$

Before we integrate to solve for $\underline{\mathbf{w}}^{(1)}$, we must remove terms independent of η in the first component of (13), terms independent of ξ in the second component of (13). These terms are premultiplied by the diagonal entries of $\mathcal{M}$.

For the linear wave equation,

$$\mathcal{M} = \begin{bmatrix} 0 & \langle k^{-1}\rangle^{-1/2}\langle \rho\rangle^{-1/2}\langle k^{-1}[\![\rho]\!]\rangle \\ \langle k^{-1}\rangle^{-1/2}\langle \rho\rangle^{-1/2}\langle \rho[\![k^{-1}]\!]\rangle & 0 \end{bmatrix}$$

(in this calculation, we have used the fact that $\langle \rho[\![k^{-1}]\!]\rangle = -\langle k^{-1}[\![\rho]\!]\rangle$, which is proved in Appendix A). Because there is no need to remove any potentially secular terms, we can integrate (13) to obtain

$$\underline{w}_1^{(1)} = \frac{n_{12}}{\lambda_1 - \lambda_2} \lambda_2^2 \underline{w}_{2\eta}^{(0)} + \underline{v}_1^{(1)}(\xi, \hat{t}), \tag{14a}$$

$$\underline{w}_2^{(1)} = \frac{n_{21}}{\lambda_2 - \lambda_1} \lambda_1^2 \underline{w}_{1\xi}^{(0)} + \underline{v}_2^{(1)}(\eta, \hat{t}), \tag{14b}$$

where $\underline{v}_1^{(1)}$ and $\underline{v}_2^{(1)}$ are integration constants.

If the diagonal entries of $\mathscr{N}$ were nonzero, we would need to eliminate some terms from (13) by introducing either $\tilde{t} = \epsilon t$ or $\tilde{x} = \epsilon x$ into our asymptotic expansion. This added degree of freedom would give us the ability to eliminate those potential secular terms by forcing $\underline{w}_1^{(0)}$ and $\underline{w}_2^{(0)}$ to satisfy two constant-coefficient diffusion equations. So now we see that $\mathscr{N}$ having zero diagonal entries means that $\tilde{t} = \epsilon t$ is absent from our asymptotic expansion and that the long-term behavior of the solution does not exhibit diffusion.

Finally, we rewrite equation (12), the x*-homogenized equation from the $\mathscr{O}(\epsilon^2)$ system, using characteristic dependent variables to obtain

$$\underline{\mathbf{w}}_t^{(0)} + \underline{\mathbf{w}}_t^{(2)} + \Lambda \underline{\mathbf{w}}_x^{(2)} = \mathscr{N} \underline{\mathbf{w}}_{tt}^{(1)} - \mathscr{R} \underline{\mathbf{w}}_{ttt}^{(0)} - \mathscr{S} \mathbf{w}_{ttx}^{(0)},$$

where

$$\mathscr{R} = P^{-1} \langle A^{-1} \rangle^{-1} \langle A^{-1} [\![A^{-1} [\![A^{-1}]\!]]\!] \rangle P,$$

and

$$\mathscr{S} = P^{-1} \langle A^{-1} \rangle^{-1} \langle A^{-1} [\![[\![A^{-1}]\!]]\!] \rangle P.$$

Using (14), and rewriting in terms of characteristic independent variables, we arrive at

$$\begin{bmatrix} (\lambda_1 - \lambda_2) w_{1\eta}^{(2)} \\ (\lambda_2 - \lambda_1) w_{2\xi}^{(2)} \end{bmatrix} = \mathscr{N} \begin{bmatrix} \frac{n_{12}}{\lambda_1 - \lambda_2} \lambda_2^4 \underline{w}_{2\eta\eta\eta}^{(0)} + \lambda_1^2 \underline{v}_{1\xi\xi}^{(1)} \\ \frac{n_{21}}{\lambda_2 - \lambda_1} \lambda_1^4 \underline{w}_{1\xi\xi\xi}^{(0)} + \lambda_2^2 \underline{v}_{2\eta\eta}^{(1)} \end{bmatrix} + \mathscr{R} \begin{bmatrix} \lambda_1^3 w_{1\xi\xi\xi}^{(0)} \\ \lambda_2^3 w_{2\eta\eta\eta}^{(0)} \end{bmatrix} - \mathscr{S} \begin{bmatrix} \lambda_1^2 w_{1\xi\xi\xi}^{(0)} \\ \lambda_2^2 w_{2\eta\eta\eta}^{(0)} \end{bmatrix}. \tag{15}$$

Before we integrate to obtain $\underline{\mathbf{w}}^{(2)}$, we need to remove all terms independent of η from the first component of (15) and all terms independent of ξ from the second component, because these will lead to secular terms. This removal produces the consistency conditions

$$\underline{w}_{1\hat{t}}^{(0)} = \left(\frac{n_{12} n_{21}}{\lambda_2 - \lambda_1} \lambda_1^4 + r_{11} \lambda_1^3 - s_{11} \lambda_1^2 \right) \underline{w}_{1\xi\xi\xi}^{(0)}$$

$$\underline{w}_{2\hat{t}}^{(0)} = \left(\frac{n_{12} n_{21}}{\lambda_1 - \lambda_2} \lambda_2^4 + r_{22} \lambda_2^3 - s_{22} \lambda_2^2 \right) \underline{w}_{2\eta\eta\eta}^{(0)}.$$

To solve these consistency conditions, it is most convenient to revert to physical independent variables. We define $\underline{\mathbf{y}}^{(0)}(x,t) = \underline{\mathbf{w}}^{(0)}(\xi,\eta,\hat{t})$ and obtain

$$\underline{y}_{1t}^{(0)} + \lambda_1 \underline{y}_{1x}^{(0)} = \epsilon^2 \left(\frac{n_{12} n_{21}}{\lambda_2 - \lambda_1} \lambda_1^4 + r_{11} \lambda_1^3 - s_{11} \lambda_1^2 \right) \underline{y}_{1xxx}^{(0)} \tag{16a}$$

$$\underline{y}_{2t}^{(0)} + \lambda_2 \underline{y}_{2x}^{(0)} = \epsilon^2 \left(\frac{n_{12} n_{21}}{\lambda_1 - \lambda_2} \lambda_2^4 + r_{22} \lambda_2^3 - s_{22} \lambda_2^2 \right) \underline{y}_{2xxx}^{(0)}. \tag{16b}$$

(To obtain (16a), we consider $\hat{t}$ and ξ as the two independent variables that are being replaced by x and t; for the second equation, $\hat{t}$ and η are the two independent variables.)

Notice that these two equations are linearized Korteweg–de Vries (KdV) equations with constant coefficients. The terms with three x derivatives indicate the dispersive nature of these equations, and the ϵ^2 premultiplying these terms shows the relative strength of the dispersive effects to the advection represented on the left-hand sides.

Given the initial conditions

$$P \underline{\mathbf{y}}^{(0)}(x, 0) = \underline{\mathbf{u}}^{(0)}(x, 0, 0) = \underline{\mathbf{h}}^{(0)}(x) = \begin{bmatrix} \sin(\pi x) \\ 0 \end{bmatrix}, \tag{17}$$

we use the dispersion relations of (16) to find the solution

$$\underline{y}_1^{(0)}(x, t) = \frac{1}{2} \sin \left[\pi (x - \lambda_1 t) - \epsilon^2 \pi^3 \left(\frac{n_{12} n_{21}}{\lambda_2 - \lambda_1} \lambda_1^4 + r_{11} \lambda_1^3 - s_{11} \lambda_1^2 \right) t \right]$$

$$\underline{y}_2^{(0)}(x, t) = -\frac{1}{2} \sin \left[\pi (x - \lambda_2 t) - \epsilon^2 \pi^3 \left(\frac{n_{12} n_{21}}{\lambda_1 - \lambda_2} \lambda_2^4 + r_{22} \lambda_2^3 - s_{22} \lambda_2^2 \right) t \right].$$

Notice that when ϵ and t are small, the solution represents the initial conditions being advected at the speeds λ_1 and λ_2. Only when $t > \mathcal{O}(\epsilon^{-2})$ do we see the dispersion having a significant effect on the wave speeds.

Using a spectral numerical technique (see Appendix B), we calculated the solution to the original Equation (5) for these initial conditions. (We did not include corrections to the initial conditions (17) discussed in Section 2.2 for the numerical solution. These corrections would only improve the correspondence between the analytic and numeric solutions.) To verify that Equations (16) describe the correct long-term behavior, we compare its solutions to the numerical solution. We also compare the numeric solution with the solution to (16) without the dispersive terms. This "nondispersive" solution is the same solution that would be obtained if we solved (5) without any slow temporal or stretched spatial scales.

Figure 1 makes these comparisons for the linear wave equation, with

$$\rho(x^*) = 1 + 0.5\cos(\pi x^*) - 0.3\sin(\pi x^*)$$

and

$$k(x^*) = \frac{1}{1 - 0.1\cos(\pi x^*) + 0.25\sin(\pi x^*)},$$

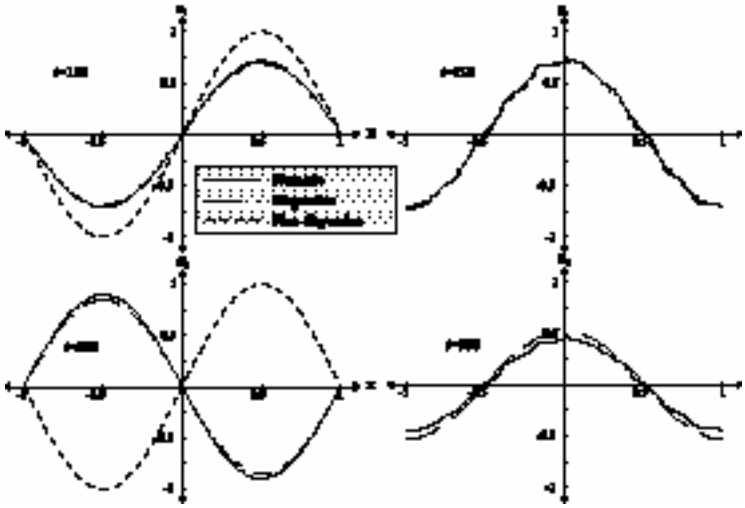


Figure 1. Numerical solution versus two analytic solutions for the linear wave equation.

and $\epsilon = 0.1$. Because the period of the medium is 0.2, and the period of the initial conditions is 2, the solution is itself periodic with period 2. Therefore, the solutions are only displayed between $-1 < x < 1$. (This periodicity also enables us to use spectral numerical techniques, which are more accurate than comparable finite difference methods.) At $t = 150$, we see the analytic solution without dispersion begins to deviate from the numeric solution. By $t = 500$, the solution is almost completely out of phase with the numeric solution. The analytic solution to (16), taking into account the dispersive term, looks very good even at $t = 500$. Because the wave speed in this problem is approximately one, by $t = 500$ almost 250 complete waves have passed through the computation domain.

To summarize, we have seen that the dispersive effect of the fluctuations in $A(x^*)$ is real, and that the multiple-scale analysis accurately describes this behavior through long-term evolution equations (consistency conditions). If we continue our analysis to higher orders of ϵ, we can determine more accurate long-term evolution equations.

3.1.6. *Solving the homogenized equations for an initial-boundary value problem.* Now, we turn our attention to the initial-boundary value problem for (5) to demonstrate the difficulty of the missing boundary conditions. Suppose that our solution domain is the quarter space $x > 0$ and $t > 0$. Because one eigenvalue of $A(x^*)$ is always positive, and the other is always negative, we can only specify one boundary condition at $x = 0$ (because it is a space-like arc):

$$u_1(0, t; \epsilon) = g(t; \epsilon) = g^{(0)}(t) + \mathcal{O}(\epsilon) \qquad \text{for} \qquad t > 0. \tag{18}$$

Also, let us prescribe the initial conditions

$$\mathbf{u}(x,0) = \underline{\mathbf{h}}^{(0)}(x) + \epsilon \mathbf{h}^{(1)}(x, x^*) + \ldots \qquad \text{for} \qquad x > 0. \tag{19}$$

Now, we try to solve the consistency conditions (16) to demonstrate the difficulty of missing boundary conditions. What sorts of initial and boundary conditions are needed to solve these equations, and how many conditions do we have?

One way to figure out what conditions are needed to solve (16) is to use a Laplace transform in t to turn these partial differential equations into a pair of third-order, constant coefficient, linear ordinary differential equations (ODEs). We can write these two ODEs in the generic form

$$sY - f(x) + \lambda Y_x = \mu Y_{xxx},$$

where $Y(x;\ s)$ is the Laplace transform of either $\underline{y}_1^{(0)}$ or $\underline{y}_2^{(0)}$, and s is the transformed variable, which must range from $\alpha - i\infty$ to $\alpha + i\infty$. (Choose α so that the integration contour is to the right of all singularities in the complex plane.) Because there is one derivative in time, we need one initial condition for each $\underline{y}_i^{(0)}$. The characteristic equation for this ordinary differential equation has three roots. If we require that the solution remains bounded as $x \to \infty$, then we need as many boundary conditions at $x = 0$ as the number of roots with nonpositive real part for the given range of s. We eliminate the possibility of all three roots having positive real part, because this would lead to the trivial solution. The two partial differential equations (16) may require different numbers of boundary conditions at $x = 0$, but each requires at least one.

Now, let us examine what conditions are available to solve (16). First, the initial condition (19) provides us with

$$\underline{\mathbf{y}}^{(0)}(x,0) = P^{-1}\underline{\mathbf{u}}^{(0)}(x,0) = P^{-1}\underline{\mathbf{h}}^{(0)}(x)$$

for $x > 0$, which is precisely what we need. Next, the boundary condition (18) gives

$$\underline{u}_1^{(0)}(0,t) = p_{11}\underline{y}_1^{(0)}(0,t) + p_{12}\underline{y}_2^{(0)}(0,t) = g^{(0)}(t)$$

for $t > 0$. Armed with only a linear combination of the boundary conditions along $x = 0$, we do not have enough information to determine a unique solution to (16). Notice that this difficulty does not occur for the initial-value problem on the infinite domain $-\infty < x < \infty$, because there is not a boundary at $x = 0$, and hence, no boundary conditions to satisfy.

3.1.7. *Reduced multiple-scales solution.* To get around the missing boundary condition difficulty, let us revisit our assumption of which scales are present in the expansion of $\mathbf{u}(x,t;\epsilon)$. Practically speaking, we use slow temporal or stretched spatial scales to avoid secular terms in our asymptotic expansion, because secular terms limit the region of validity of our solution. Adding these

slow scales gives us the freedom to eliminate potentially secular terms by enforcing consistency conditions. However, at the moment, we are unable to solve these consistency conditions because of a lack of boundary information. Without slow temporal or stretched spatial scales, we cannot avoid secular terms, but at least a solution valid in the near field can be found. From this near-field solution we can extract the "missing" boundary conditions that we need to solve the consistency conditions.

Let us illustrate this procedure for (5). We re-expand the state variables using the reduced set of scales, x^*, x, and t:

$$\mathbf{u}(x,t;\epsilon) = \underline{\mathbf{r}}^{(0)}(x,t) + \epsilon \mathbf{r}^{(1)}(x,x^*,t) + \ldots.$$

We use $\mathbf{r}^{(i)}$ instead of $\mathbf{u}^{(i)}$ to avoid confusion with the previous multiple-scale expansion of $\mathbf{u}(x,t;\epsilon)$. It is unnecessary to repeat the work of obtaining the equations for each order of ϵ, because the only change is that there are no derivatives with respect to any slow temporal scales. Notice that by writing $\mathbf{r}^{(0)}(x,t)$ instead of $\mathbf{r}^{(0)}(x,x^*,t)$ in our new expansion, we have skipped the step of using the $\mathcal{O}(\epsilon^{-1})$ system to eliminate the x^*-dependence of the leading-order solution.

Following the steps outlined in Section 3.1.2, we define the characteristic independent variables $\xi = x - \lambda_1 t$ and $\eta = x - \lambda_2 t$, along with the characteristic dependent variables $\underline{\mathbf{r}}^{(0)}(x,\ t) = P\underline{\mathbf{s}}^{(0)}(\xi,\ \eta)$. As in Section 3.1.5, the governing equations for $\underline{\mathbf{s}}^{(0)}$ are $\underline{s}_{1\eta}^{(0)} = 0$ and $\underline{s}_{2\xi}^{(0)} = 0$, which imply that

$$\underline{\mathbf{s}}^{(0)}(\xi,\eta) = \begin{bmatrix} \underline{s}_1^{(0)}(\xi) \\ \underline{s}_2^{(0)}(\eta) \end{bmatrix}.$$

In contrast to the analysis of the previous section, $\underline{\mathbf{s}}^{(0)}$ can be completely determined at this stage, because we do not allow it to depend on $\hat{t}$. Using the $\mathcal{O}(1)$ initial conditions from (19),

$$\underline{\mathbf{s}}^{(0)}(x,x) = \begin{bmatrix} \underline{\mathbf{s}}_1^{(0)}(x) \\ \underline{\mathbf{s}}_2^{(0)}(x) \end{bmatrix} = P^{-1}\underline{\mathbf{r}}^{(0)}(x,0) = P^{-1}\underline{\mathbf{h}}^{(0)}(x) \qquad \text{for} \qquad x > 0.$$

Therefore,

$$\underline{s}_1^{(0)}(\xi) = \frac{1}{\det P}\left[p_{22}\underline{h}_1^{(0)}(\xi) - p_{12}\underline{h}_2^{(0)}(\xi)\right] \qquad \text{for} \qquad \xi > 0$$

$$\underline{s}_2^{(0)}(\eta) = -\frac{1}{\det P}\left[p_{21}\underline{h}_1^{(0)}(\eta) - p_{11}\underline{h}_2^{(0)}(\eta)\right] \qquad \text{for} \qquad \eta > 0.$$

We need not worry about the determinant of P being zero, because P is nonsingular.

Because $\eta = x - \lambda_2 t$ and $\lambda_2 < 0$, the second state variable $\underline{s}_2^{(0)}(\eta)$ propagates information to the "left" in the x-t plane. Furthermore, η is always positive in the solution domain so $\underline{s}_2^{(0)}(\eta)$ is completely determined. On the other hand, $\xi = x - \lambda_1 t$ and $\lambda_1 > 0$, so ξ can take on positive and negative values in the solution domain. The first state variable $\underline{s}_1^{(0)}(\xi)$ propagates information to the "right" so the initial conditions only determine $\underline{s}_1^{(0)}(\xi)$ for $\xi > 0$. We still have to define $s_1{}^{(0)}(\xi)$ for $\xi < 0$ using the $\mathcal{O}(1)$ boundary condition (18). Applying equation (18) gives $g^{(0)}(t) = \underline{r}_1^{(0)}(0, t) = p_{11}\underline{s}_1^{(0)}(-\lambda_1 t) + p_{12}\underline{s}_2^{(0)}(-\lambda_2 t)$ for $t > 0$. Therefore,

$$\underline{s}_1^{(0)}(\xi) = \frac{1}{p_{11}} g^{(0)}(-\xi/\lambda_1) + \frac{p_{12}}{p_{11}\det P}\left(p_{21}\underline{h}_1^{(0)}(\xi\lambda_2/\lambda_1) - p_{11}\underline{h}_2^{(0)}(\xi\lambda_2/\lambda_1)\right) \quad \text{for} \quad \xi < 0.$$

We are now able to recover the "missing" boundary condition,

$$\underline{r}_2^{(0)}(0,t) = \frac{p_{21}}{p_{11}}\left[g^{(0)}(t) - \underline{h}_1^{(0)}(-\lambda_2 t)\right] + \underline{h}_2^{(0)}(-\lambda_2 t).$$

If necessary, we can even calculate quantities such as $\underline{r}_{2x}^{(0)}(0, t)$ by taking a derivative of $\underline{\mathbf{r}}^{(0)}$ with respect to x, then setting $x = 0$.

The only potential difficulty in the analysis above is if p_{11} vanishes. We digress briefly to show why p_{11} cannot be zero. We can show that if $p_{11} = 0$, then $\alpha_{12} = 0$ as well, where α_{ij} is the i-j entry of $\langle \mathrm{A}^{-1}\rangle^{-1}$. Furthermore, by our choice of $\lambda_1 > 0$ and $\lambda_2 < 0$, we can also infer that $\alpha_{11} = \lambda_2 < 0$. These facts imply that

$$\underline{r}_{1t}^{(0)} + \alpha_{11}\underline{r}_{1x}^{(0)} = 0,$$

or that the first component of the leading-order homogenized solution advects information to the left in the x-t plane with speed $|\lambda_2|$. That means that we cannot specify a boundary condition on the first component, $u_1(0,t;\epsilon)$. Therefore, we must assume $p_{11} \neq 0$ to avoid this nonsensical result.

The procedure outlined above for recovering boundary conditions can be repeated for the equations that arise at higher orders of ϵ. Solving the $\mathcal{O}(\epsilon)$ system of equations using the reduced set of scales x^*, x, and t, enables us to find the missing boundary condition $r_2{}^{(1)}(0,0,t)$. This process will not be demonstrated here, because the algebra grows exponentially with each order of ϵ. We have developed *Mathematica* notebooks that are capable of performing this tedious task to any order of ϵ.

For example, suppose that for the linear wave Equation (4), with the specific functions

$$\rho(x^*) = 1 + 0.5\cos(\pi x^*) - 0.3\sin(\pi x^*)$$

and

$$k(x^*) = \frac{1}{1 - 0.1\cos(\pi x^*) + 0.25\sin(\pi x^*)},$$

we impose the boundary condition

$$u_1(0, t; \epsilon) = g^{(0)}(t) = 1 - \cos t.$$

In addition, suppose that the initial conditions are

$$\mathbf{h}(x; \epsilon) = \begin{bmatrix} \sin\pi x \\ 0 \end{bmatrix} - \epsilon \begin{bmatrix} \cos(\pi x)[0.1\sin(\pi x^*) + 0.25\cos(\pi x^*)] \\ 0 \end{bmatrix} + \mathcal{O}(\epsilon^2),$$

where the fluctuating parts are chosen so that the solution does not depend on $t^* = t/\epsilon$ (see Section 2.2). Using *Mathematica*, we calculate the missing boundary condition to be

$$\begin{aligned} u_2(0, t; \epsilon) = 1 - \cos(t) - \sin(\pi t) + \epsilon \left[\frac{\cos(\pi t)}{4} - \frac{201\sin t}{400\pi} \right] \\ + \epsilon^2 \left[\frac{52,639\pi t \cos(\pi t)}{320,000} + \frac{399,361\cos t + 209,161\pi^2 \sin(\pi t)}{320,000\pi^2} \right] \\ + \mathcal{O}(\epsilon^3). \end{aligned} \tag{20}$$

Notice that the $\mathcal{O}(\epsilon^2)$ contribution to this recovered boundary information contains a secular term proportional to t. The presence of this secular term makes the expansion valid only for t roughly less than ϵ^{-1}. As we will see, the region of validity is, in practice, usually larger.

Because we do not have an exact solution to the original linear wave equation, we must verify (20) numerically. Using CLAWPACK, we generate a numerical approximation to the solution of the original linear wave equation (5) with $A(x^*)$ defined in (6), and extract the values of $u_2(0,t)$ to compare with (20). (See Appendix B for more discussion about CLAWPACK.) Because there is no computer large enough to represent the semi-infinite solution domain $x > 0$ using finite volumes, we limit ourselves to a finite computation domain of $0 < x < L$. The upper limit L is chosen so that a wave entering the computation domain from the right boundary does not interfere with the part of the solution that we desire. We choose $L = 110$, because we are interested in calculating $u_2(0,t)$ up to $t = 100$ and in this problem, the wave speeds are ± 1. At $x = 0$, we specify $u_1(0,t)$ using the given boundary condition, and simulate a free condition on $u_2(0,t)$ using zero-order extrapolation. We also use zero-order extrapolation for both u_1 and u_2 at $x = L$.

For $\epsilon = 0.1$, we find that the leading-order term of (20) matches the numerically calculated $u_2(0, t)$ very well for $t < 10$. As t gets larger, we must take into account the higher-order corrections in (20) to get a better correspondence. Figure 2 compares the analytically recovered boundary information against the numeric solution for $75 < t < 100$. In pane (a), we compare the numeric solution with the $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ terms of (20) only; in pane (b), we include all terms displayed in (20). Notice that without the $\mathcal{O}(\epsilon^2)$ contribution, the analytic boundary information does not quite match the numeric solution. The contribution is, therefore, useful for $t < 100$, although it introduces a secular term.

A convincing way to verify that an expansion is asymptotically correct is to plot the absolute error of the expansion against ϵ using logarithmic scales [10]. The slope of the resulting line indicates the rate of convergence of the asymptotic solution to the numeric solution as $\epsilon \to 0$. Figure 3 shows the absolute error of the analytically recovered boundary information (20), with different terms of the asymptotic expansion included. The absolute error is measured using the discretized version of the integral

$$\text{error} = \int_0^T \left| u_2^{(\text{n})}(0, t) - u_2^{(\text{a})}(0, t) \right| dt,$$

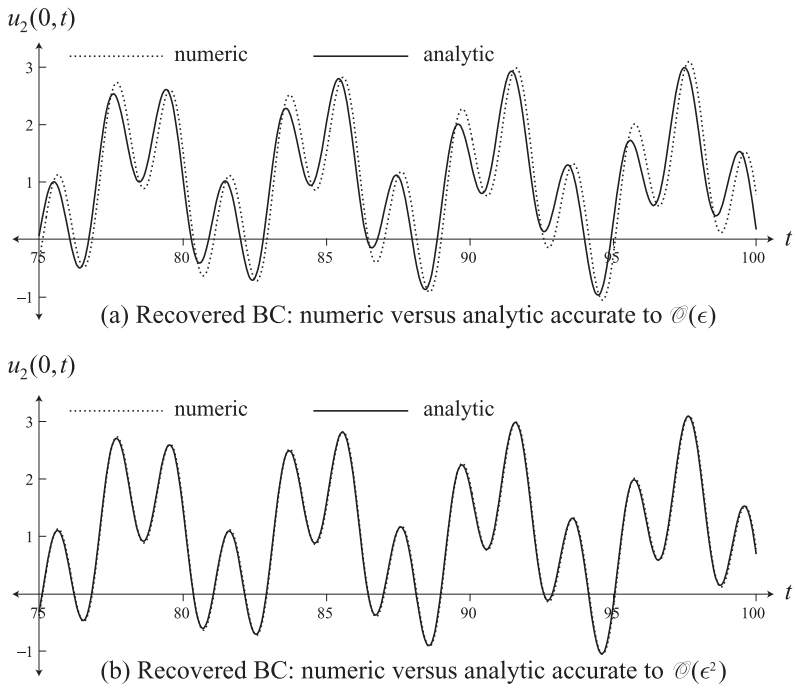


Figure 2. Numeric verification of the recovered boundary information for an initial boundary value problem for the linear wave equation.

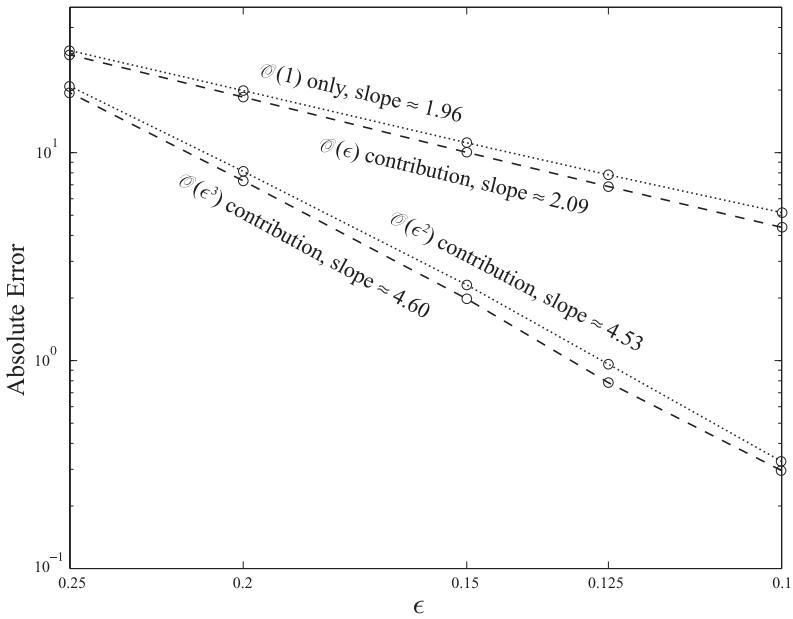


Figure 3. Rate of convergence of analytic recovered boundary information for an initial boundary value problem for the linear wave equation.

where $u^{(n)}$ is the numeric solution, and $u^{(a)}$ is the analytic expression that we are observing. Ideally, we would compare the asymptotic expansion against the exact solution, but because an exact solution is not available, we must make do with a numerically calculated solution. Again, we use CLAWPACK for our numeric solution, but this time with a spatial domain of $0 < x < 60$ and a spatial step size of 0.002. We, then, sample the boundary information, $u_2(0,t)$, in the range $0 < t < 50$ using 2,000 points.

Figure 3 shows a surprising result. Keeping only the leading order term in (20), we would normally expect the truncated error to be $\mathcal{O}(\epsilon)$ as $\epsilon \to 0$. However, we see that the absolute error of the leading-order term is much smaller and actually behaves more like ϵ^2 as $\epsilon \to 0$. This finding does not invalidate the asymptotic correctness of (20); on the contrary, it just means that the leading-order term is more accurate than expected. Similarly, we see that the asymptotic expansion including the $\mathcal{O}(\epsilon^2)$ contribution converges to the true solution slightly faster than ϵ^3 as $\epsilon \to 0$, which we would normally expect. The asymptotic expansion including up to the $\mathcal{O}(\epsilon)$ contribution only and the expansion with contributions up to $\mathcal{O}(\epsilon^3)$ (not shown) converge close to their expected rates.

Now, let us also recover the missing boundary conditions for a case in which the initial conditions are zero (a signaling problem). We impose the same boundary condition

$$u_1(0, t; \epsilon) = g^{(0)}(t) = 1 - \cos t,$$

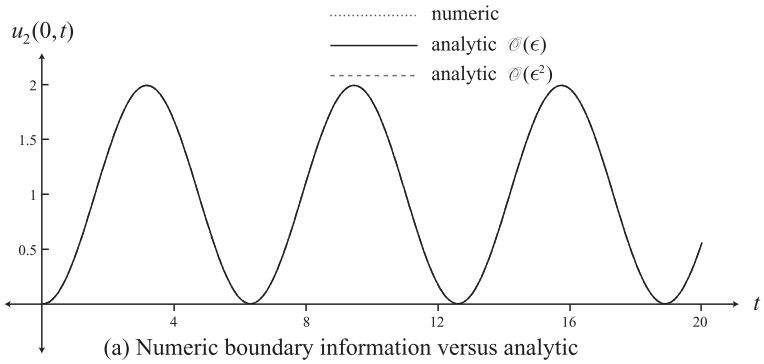
but instead, specify that $\mathbf{u}(x,0;\epsilon) = \mathbf{0}$. Under these conditions, we calculate the missing boundary condition to be

$$u_2(0,t;\epsilon) = 1 - \cos t - \epsilon\frac{201\sin t}{400\pi} + \epsilon^2\frac{399,361\cos t}{320,000\pi^2} + \mathcal{O}(\epsilon^3). \tag{21}$$

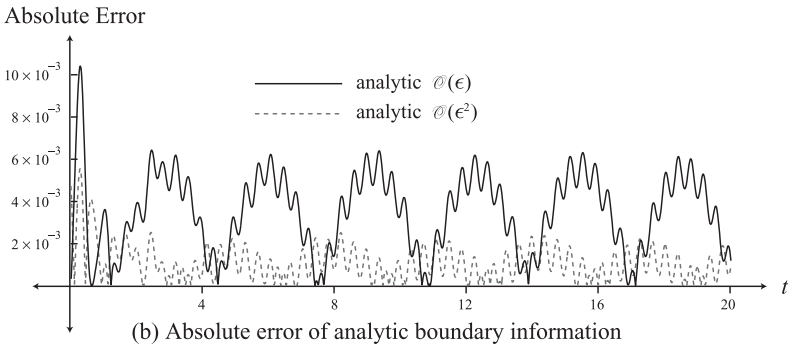
When we compare (21) to the recovered boundary information with nonzero initial conditions (20), we see that there are no secular terms in the recovered boundary information for the signaling problem.

Figure 4a shows the analytically recovered boundary information in (21) against its numeric counterpart, calculated solution using $\epsilon = 0.2$ and a spatial step size of 0.0005 for $0 < t < 20$. (We used $0 < x < 25$ as our computational domain.) Displayed in this manner, the numeric and analytic $u_2(0,t)$ are indistinguishable. Figure 4b shows that the $\mathcal{O}(\epsilon^2)$ contribution to the recovered boundary condition does, indeed, increase its accuracy.

Once again, we compare the recovered boundary information (21) against the numeric solution for various ϵ and plot the absolute error versus ϵ using a logarithmic scale. Figure 5 shows that the leading-order term of (21) by itself converges to the true boundary information like ϵ as $\epsilon \to 0$, as expected.



(a) Numeric boundary information versus analytic



(b) Absolute error of analytic boundary information

Figure 4. Numeric verification of the recovered boundary information for a signaling problem for the linear wave equation.

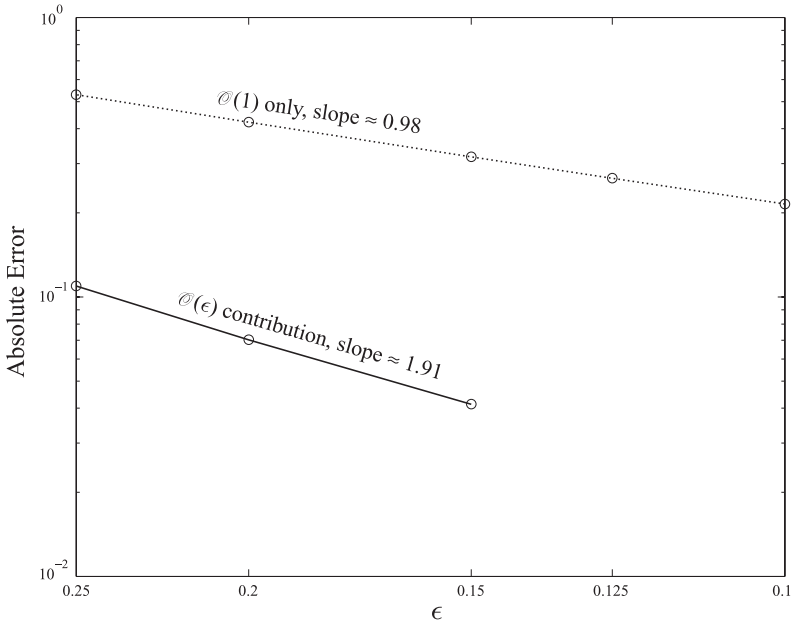


Figure 5. Rate of convergence of analytic recovered boundary information for a signaling problem for the linear wave equation.

Adding the $\mathcal{O}(\epsilon)$ contribution in (21) accelerates this convergence to ϵ^2 as $\epsilon \to 0$. We were unable to verify this latter result for the same range of ϵ, because the numeric calculation reached its limit of accuracy for our chosen spatial step size. As a result, we were also unable to verify that including higher-order terms in (21) improved the agreement between the analytic and numeric solutions. Reducing the spatial step size would enable us to verify these claims (because it would increase the over-all accuracy of the numerical method and better resolve the fluctuations of the $A(x^*)$ matrix), but it would also increase computation time.

Now that we have seen how to use a reduced set of multiple scales to recover boundary information that is asymptotically correct, we turn to the most striking feature of (20): the presence of secular terms proportional to t. What is the mechanism that causes these secular terms to appear in the recovered boundary information? By neglecting slow temporal and stretched spatial scales, secular terms (proportional to x and t) will arise in the solution. Therefore, it is not surprising that secular terms proportional to t may remain after we substitute $x = 0$.

What is more interesting is why the secular terms go away when the initial conditions are zero. Using *Mathematica*, we have calculated the recovered boundary information to high orders of ϵ. For the linear wave equation (4), there is only one way for the boundary information to be free

of secular terms. For convenience, let us normalize $\rho(x^*)$ and $k(x^*)$ so that $\langle\rho\rangle = \langle 1/k\rangle = 1$, so the wave speeds are $\lambda_{1,2} = \pm 1$. In this case, only if $\underline{h}_1^{(0)}(x) = \underline{h}_2^{(0)}(x)$ for all $x > 0$ will the recovered boundary information be free of secular terms.

If we return to the reduced multiple-scale analysis introduced in this section, we see that the leading-order behavior of the solution is $\underline{\mathbf{r}}^{(0)}(x, t) = P\underline{\mathbf{s}}^{(0)}(\xi,\eta)$, where $\underline{s}_1^{(0)} = \underline{s}_1^{(0)}(\xi)$, and $\underline{s}_2^{(0)} = \underline{s}_2^{(0)}(\eta)$ are the characteristic dependent variables, and $\xi = x - t$ and $\eta = x + t$ are the characteristic independent variables. Because of our choice of variables, $\underline{s}_1^{(0)}$ represents a wave traveling to the "right" in the x-t plane, and $\underline{s}_2^{(0)}$ represents a wave traveling to the "left." With ρ and k normalized, $\underline{s}_2^{(0)}(\eta) = \underline{h}_2^{(0)}(\eta) - \underline{h}_1^{(0)}(\eta)$ for all $\eta > 0$, so if $\underline{h}_1^{(0)}(x) = \underline{h}_2^{(0)}(x)$ then $\underline{s}_2^{(0)} = 0$. *Therefore, we see that the recovered boundary information is free of secular terms only when the wave traveling to the left in the x-t plane is identically zero.*

Why would the recovered boundary information be free of secular terms when the initial conditions are such that the left-going wave vanishes? The reason is that without the slow and stretched scales, the multiple-scale analysis produces a solution that models advection only. This purely advected solution is only accurate for small t, because the true long-term behavior of the solution includes dispersion and other higher-order effects. The recovered boundary information is the result of the interaction between the partial differential equation, the given boundary condition, and the left-going wave hitting the boundary. If the left-going wave is nonzero, the recovered boundary information will become inaccurate for large t. Therefore, the presence of secular terms is merely a sign that the recovered boundary information cannot be trusted for large t. Neglecting these secular terms not only makes the recovered boundary information less accurate (as in the example initial-boundary value problem in this section), it also gives the false impression that the recovered boundary information can be used for large t. However, the reader should keep in mind that for the signaling problem (initial conditions equal to zero), the left-going wave is zero, and the recovered boundary information will have no secular terms. We take advantage of this fact in the next section.

3.1.8. *Multiple-scales solution revisited.* Now armed with the recovered boundary information, we can return to solving the consistency conditions (16). A general solution to these equations would be very complicated, so we demonstrate the solution procedure for the linear wave equation with

$$\rho(x^*) = 1 + 0.5\cos(\pi x^*) - 0.3\sin(\pi x^*)$$

and

$$k(x^*) = \frac{1}{1 - 0.1\cos(\pi x^*) + 0.25\sin(\pi x^*)},$$

the initial conditions

$$\mathbf{u}(x,0;\epsilon) = P^{-1}\underline{\mathbf{h}}^{(0)} + \mathcal{O}(\epsilon) = \begin{bmatrix} \frac{1}{2}\sin(\pi x) \\ -\frac{1}{2}\sin(\pi x) \end{bmatrix} + \mathcal{O}(\epsilon),$$

and the boundary information

$$u_1(0,t;\epsilon) = g(t;\epsilon) = 1 - \cos t + \mathcal{O}(\epsilon).$$

Because the original partial differential equation (5) is linear, we can use superposition to partition the initial-boundary value problem into two problems: an initial-value problem, and a signaling problem. Let $\mathbf{u}(x,t;\epsilon) = \mathbf{u}^{(A)}(x,t;\epsilon) + \mathbf{u}^{(B)}(x,t;\epsilon)$, where $\mathbf{u}^{(A)}$ and $\mathbf{u}^{(B)}$ satisfy the same partial differential equation (5), but $\mathbf{u}^{(A)}$ satisfies the initial-value problem in the infinite domain, and $\mathbf{u}^{(B)}$ satisfies the signaling problem in the semi-infinite domain. Specifically, $\mathbf{u}^{(A)}(x,0;\epsilon) = \mathbf{h}(x;\epsilon)$ on $-\infty < x < \infty$, while $u_1^{(B)}(0,t;\epsilon) = g(t;\epsilon) - u_1^{(A)}(0,t;\epsilon)$ for $t > 0$ and $\mathbf{u}^{(B)}(x,0;\epsilon) = \mathbf{0}$ for $x > 0$. It does not matter how the initial conditions $\mathbf{h}(x;\epsilon)$ are extended for negative x, because once the solution to the initial-value problem is obtained, $u_1^{(A)}(0,t;\epsilon)$ is subtracted from the boundary information, $u_1^{(B)}(0,t;\epsilon)$. The sum of $\mathbf{u}^{(A)}$ and $\mathbf{u}^{(B)}$ will satisfy all of the original initial and boundary conditions.

There are two reasons we want to consider a signaling problem instead of a problem with nontrivial initial and boundary conditions. The first is that we have seen that the recovered boundary conditions contain secular terms when the initial conditions are nontrivial. To maximize the region of validity of our asymptotic expansion, we should use the recovered boundary conditions for the signaling problem instead. The second reason is that solving the long-term evolution equations is much simpler with zero initial conditions.

The initial-value problem for $\mathbf{u}^{(A)}$ has already been solved in Section 3.1.5. If we extend the initial conditions in the natural way for negative x, the solution for our particular choice of $\epsilon = 0.1$, $\rho(x^*)$ and $k(x^*)$ is

$$\mathbf{u}^{(A)}(x,t;\epsilon) = \frac{1}{2}\begin{bmatrix} \sin\left(\frac{31{,}947{,}361t}{32{,}000{,}000} + \pi x\right) - \sin\left(\frac{31{,}947{,}361\pi t}{32{,}000{,}000} - \pi x\right) \\ -\sin\left(\frac{31{,}947{,}361t}{32{,}000{,}000} + \pi x\right) - \sin\left(\frac{31{,}947{,}361\pi t}{32{,}000{,}000} - \pi x\right) \end{bmatrix} + \mathcal{O}(\epsilon).$$

Now we are ready to solve the signaling problem for $\mathbf{u}^{(B)}$. Let $\mathbf{u}^{(B)}(x,\, t;\, \epsilon) = P\underline{\mathbf{y}}^{(B)}(x,\, t) + \mathcal{O}(\epsilon)$, and $\underline{\mathbf{y}}^{(B)}$ satisfy the long-term evolution equations (16). The boundary conditions for $\underline{\mathbf{y}}^{(B)}(0,t)$ come from the recovered boundary conditions

for the signaling problem (21) and $\mathbf{u}^{(A)}(0,t;\epsilon)$:

$$\underline{\mathbf{y}}^{(B)}(0,t) = P^{-1}\mathbf{u}^{(B)}(0,t;0) = P^{-1}\begin{bmatrix} g^{(0)}(t) - u_1^A(0,t;0) \\ 1 - \left(\frac{399{,}361}{32{,}000{,}000\pi^2} - 1\right)\cos t - \frac{201}{4{,}000\pi}\sin t \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \left(\frac{399{,}361}{64{,}000{,}000\pi^2} - 1\right)\cos t - \frac{201}{8000\pi}\sin t \\ \frac{399{,}361\cos t - 1{,}608{,}000\sin t}{64{,}000{,}000\pi^2} \end{bmatrix}.$$

To solve for $\underline{\mathbf{y}}^{(B)}$, we use a Laplace transform in t to turn the partial differential equations (16) into third-order ordinary differential equations:

$$-\frac{52,639}{32,000,000\pi^2}\frac{d^3\underline{Y}_1^{(B)}}{dx^3} - \frac{d\underline{Y}_1^{(B)}}{dx} - s\underline{Y}_1^{(B)} = 0 \tag{22a}$$

$$\frac{52,639}{32,000,000\pi^2}\frac{d^3\underline{Y}_2^{(B)}}{dx^3} + \frac{d\underline{Y}_2^{(B)}}{dx} - s\underline{Y}_1^{(B)} = 0 \tag{22b}$$

where

$$\underline{Y}_i^{(B)}(x,s) = \mathscr{L}\left[\underline{y}_i^{(B)}(x,t)\right] = \int_0^\infty e^{-st}\underline{y}_i^{(B)}(x,t)dt.$$

Using *Mathematica*, we have verified that only one of the three roots of the characteristic equation for $\underline{Y}_1^{(B)}$ has nonpositive real part, and two of the roots of the characteristic equation for $\underline{Y}_2^{(B)}$ have nonpositive real part. Let these roots be $\sigma_1^{(1)}$ and $\sigma_1^{(2)}$, $\sigma_2^{(2)}$, respectively; we do not display these roots here for brevity. We do not use the homogeneous solutions based on the roots with positive real parts, because we want the solution to remain bounded as $x \to \infty$. The fact that there are two roots with nonpositive real part for $\underline{Y}_2^{(B)}$ means that we need two boundary conditions to find a unique solution for $\underline{Y}_2^{(B)}$. In the previous section, we explained how to obtain the extra information that we need, $\underline{y}_{2x}^{(B)}(0,t)$. We do not display this recovered boundary information to save space.

Once we have recovered all the necessary boundary conditions, the solutions to the ordinary differential equations (22a) and (22b) are

$$\underline{Y}_1^{(B)}(x,t) = \mathscr{L}\left[\underline{y}_1^{(B)}(0,t)\right]\exp\left(\sigma_1^{(1)}x\right)$$

$$\underline{Y}_2^{(B)}(x,t) = \mathscr{L}\left[\underline{y}_2^{(B)}(0,t)\right] \frac{\sigma_1^{(2)}\exp\left(\sigma_2^{(2)}x\right) - \sigma_2^{(2)}\exp\left(\sigma_1^{(2)}x\right)}{\sigma_1^{(2)} - \sigma_2^{(2)}}$$

$$+ \mathscr{L}\left[\underline{y}_{2x}^{(B)}(0,t)\right] \frac{\exp\left(\sigma_1^{(2)}x\right) - \exp\left(\sigma_2^{(2)}x\right)}{\sigma_1^{(2)} - \sigma_2^{(2)}}.$$

The difficult part is now the inversion of these expressions. Because the roots $\sigma_1^{(1)}$, $\sigma_1^{(2)}$, and $\sigma_2^{(2)}$ involve cube and square roots of s, the calculation of the Laplace inverse involves branch cuts. As a result, the inversion integrals cannot be expressed in closed form and must be approximated either analytically or numerically.

One way to make progress with an analytic approximation is to use residue calculus and Watson's lemma (see Section 2.1 and Chapter 4 of [11]). In particular, we use the result that if $F(s)$ has a branch point at s_0 and can be expressed there as

$$F(s) = (s - s_0)^{\gamma} \sum_{n=0}^{\infty} a_n (s - s_0)^n$$

with $a_0 \neq 0$ and $\gamma > -1$, then

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} F(s) e^{ts} ds \approx -\frac{e^{ts_0}\sin(\gamma\pi)}{\pi t^{\gamma+1}} \sum_{n=0}^{\infty} a_n (-1)^n t^{-n} \times \Gamma(\gamma + n + 1)$$

as $t \to \infty$. In many cases, the dominant term of this asymptotic expansion is sufficient for large t.

We locate and expand about the branch points of $\underline{Y}_1^{(B)}$ and $\underline{Y}_2^{(B)}$. Residue calculus and the dominant term of the branch point expansion give

$$\underline{y}_1^{(B)}(x,t) \approx 1 - 0.999368\cos(t - 1.00017x) - .00799754\sin(1 - 1.00017x)$$

$$+ xt^{-3/2}[6.81765 \times 10^{-5}\cos(29.8139t - 44.7209x)$$

$$- 2.01145 \times 10^{-5}\sin(29.8193t - 44.7209x)] \qquad (23)$$

Note that the first two terms of (23) are the most significant and that they model the advection of the boundary information, $\underline{y}_1^{(B)}(0,t)$, with a small amount of dispersion. We do not show $\underline{y}_2^{(B)}$, because it is too long. The coefficients in the expansions of $\underline{Y}_1^{(B)}$ and $\underline{Y}_2^{(B)}$ about its branch points involve increasing powers of x, because we are essentially seeking expansions of $\exp(\sigma x)$. If x is large, and the real part of σ is negative, the power series expansion of $\exp(\sigma x)$ requires

many terms before the series begins to converge. This means that the approximation given above is only valid for large t and small x.

3.1.9. *Summary and discussion.* We have seen that the cumulative (long-term) effects of the x^*-scale fluctuations in the linear wave equation only appear on the $\hat{t} = \epsilon^2 t$ scale. This fact is a direct consequence of the diagonal entries of $\mathcal{M}$ being zero, and it implies that the long-term evolution of the solution does not include diffusion.

We believe that it is more than coincidence that diffusion should be absent for solutions to remain bounded. The reader should keep in mind that the dependent variables **u** come from a perturbation to a steady-state solution for an arbitrary conservation law. Therefore, if **u** grows in time, its corresponding steady-state solution is unstable. We believe that for all physically realistic situations in which the steady-state solution is stable (implying that **u** remains bounded), the homogenized solution should not diffuse on the $\tilde{t}$ or $\tilde{x}$ scales.

As we see in the next section, the presence of diffusion depends solely on the matrix $A(x^*)$. With some algebra, we can compute that diffusion will be absent if

$$\langle \alpha_{12}[\![\alpha_{21}]\!]\rangle\langle \alpha_{11} - \alpha_{22}\rangle + \langle (\alpha_{11} - \alpha_{22})[\![\alpha_{12}]\!]\rangle\langle \alpha_{21}\rangle + \langle a_{21}[\![\alpha_{11} - \alpha_{22}]\!]\rangle\langle \alpha_{12}\rangle = 0, \tag{24}$$

where $\alpha_{ij}(x^*)$ is the *i-j* entry of $A^{-1}(x^*)$. Note that this condition is easily satisfied if either $\alpha_{11} = \alpha_{22}$ or $\alpha_{12} = \alpha_{21}$. (The latter fact and the computation of this condition require the fact that $\langle a[\![b]\!]\rangle = -\langle b[\![a]\!]\rangle$, which is proved in Appendix A.) In all of the physical examples we have seen, (24) is satisfied. (In [9], we show that a nonphysical problem devised by Kevorkian and Bosley in [1] violates (24) at the expense of having exponentially growing solutions, suggesting that (24) may be useful for determining the stability of steady-state solutions used to produce problems of the form (1).)

Assuming that diffusion is absent from the consistency conditions, the next observable long-term phenomenon is dispersion. Problems that exhibit dispersion and not diffusion have a pair of decoupled, linear KdV equations as their long-term evolution equations (consistency conditions). For the solution domain $x > 0$ and $t > 0$, these PDEs require one or two boundary conditions along $x = 0$ and one initial condition along $t = 0$ for each dependent variable. Because our original hyperbolic problem only allows us to specify a linear combination of both state variables along the boundary $x = 0$, we cannot solve the long-term evolution equations armed with only the information necessary for a well-posed configuration of the original problem (5).

In Section 3.1.7, we showed that this problem can be circumvented by temporarily ignoring the dependence of the solution on any slow temporal

scales during the multiple-scale analysis. Without these scales, we avoid those problematic consistency conditions at the expense of a solution that is asymptotically valid in a smaller region of the solution domain. Once this near-field solution has been obtained, we can extract the "missing" boundary conditions and return to the original multiple-scale homogenization technique.

We have also seen that the recovered boundary information will contain secular terms proportional to t when the initial conditions are nontrivial. The presence of these secular terms restricts the region of validity of the calculated solution. Fortunately, the signaling problem is one situation in which the recovered boundary information does not contain secular terms. In Section 3.1.8, we employed this fact to our advantage by using superposition to partition the problem into two pieces, an initial-value problem on the semi-infinite domain, and a signaling problem with zero initial conditions. The initial-value problem is solved using multiple scales, including slow temporal scales. The consistency conditions for the signaling problem are solved using the recovered boundary information from a multiple-scale analysis with a reduced set of scales. In this manner, we are able to construct a solution that is asymptotically valid for a long time.

3.2. The general case

We now turn our attention to the general nonlinear problem represented by

$$\mathbf{u}_t + A(x^*)\mathbf{u}_x + B(x^*)\mathbf{u} = \epsilon[C(\mathbf{u}, x^*)\mathbf{u} + D(\mathbf{u}, x^*)\mathbf{u}_x] + \mathcal{O}(\epsilon^2), \tag{25}$$

which we introduced in Section 2.1. Recall that (25) arises when we linearize a general system of conservation laws about its steady-state solution. In this section, we discuss the solution of (25) when $\mathbf{u}$ is a vector of two unknown functions and when both initial and boundary conditions are present.

Suppose the initial conditions for (25) are

$$\mathbf{u}(x, 0; \epsilon) = \underline{\mathbf{h}}^{(0)}(x) + \epsilon \mathbf{h}^{(1)}(x, x^*) + \mathcal{O}(\epsilon^2), \tag{26}$$

where the fluctuating parts of higher-order terms cannot be specified arbitrarily if we want to ensure a solution that is independent of the fast time variable, $t^* = t/\epsilon$ (see Section 2.2).

Equation (25) is a system of first-order partial differential equations, and its leading order behavior is governed by the linear terms on the left-hand side. Following the discussion in Section 2.1, we require $A(x^*)$ to be invertible and further assume that one eigenvalue is always positive and the other always negative. This assumption allows us to specify only one boundary condition at $x = 0$. The most general way of writing this boundary condition is to specify a linear combination of the state variables,

$$\alpha u_1(0, t; \epsilon) + \beta u_2(0, t; \epsilon) = g(t; \epsilon) = g^{(0)}(t) + \mathcal{O}(\epsilon), \tag{27}$$

for $t > 0$. However, we do not lose any generality by assuming that $\alpha = 1$ and $\beta = 0$. In other words, we may assume that the first dependent variable is specified at the boundary $x = 0$, and the second dependent variable is left unspecified. The reason we can do this is that for any constant 2×2, nonsingular matrix Z, the substitution $\mathbf{u} = Z\mathbf{v}$ does not fundamentally change the form of our problem. This substitution changes (25) to

$$\mathbf{v}_t + Z^{-1}A(x^*)Z\mathbf{v}_x + Z^{-1}B(x^*)Z\mathbf{v} = \epsilon[Z^{-1}C(Z\mathbf{v}, x^*)Z\mathbf{v} + Z^{-1}D(Z\mathbf{v}, x^*)Z\mathbf{v}_x] + \mathcal{O}(\epsilon^2),$$

that is still of the form represented by (25). However, the substitution changes the boundary condition (27) to

$$[\alpha\ \beta]Z\mathbf{v}(0,t) = g^{(0)}(t) + \epsilon g^{(1)}(t) + \cdots,$$

so we can use it to alter the linear combination of $u_1(0,t;\epsilon)$ and $u_2(0,t;\epsilon)$.

Suppose the solution to (25) has the asymptotic expansion

$$\mathbf{u}(x,t;\epsilon) = \mathbf{u}^{(0)}(x,x^*,\tilde{x},t,\tilde{t}) + \epsilon\mathbf{u}^{(1)}(x,x^*,\tilde{x},t,\tilde{t}) + \mathcal{O}(\epsilon^2).$$

We include the slow and stretched scales $\tilde{x} = \epsilon x$ and $\tilde{t} = \epsilon t$, because we only intend to carry out the analysis to the $\mathcal{O}(\epsilon)$ system of equations—if we want to continue the analysis to $\mathcal{O}(\epsilon^2)$, we must also include the $\mathcal{O}(\epsilon^2)$ contribution in (25). However, the solution method does not change significantly by including more spatial or temporal scales.

As before, the multiple-scale analysis proceeds by plugging into (25) the expressions

$$\mathbf{u}_t \to \mathbf{u}_t^{(0)} + \epsilon(\mathbf{u}_t^{(1)} + \mathbf{u}_{\tilde{t}}^{(0)}) + \ldots$$

and

$$\mathbf{u}_x \to \epsilon^{-1}\mathbf{u}_{x^*}^{(0)} + (\mathbf{u}_x^{(0)} + \mathbf{u}_{\mathbf{x}^*}^{(\mathbf{1})}) + \epsilon(\mathbf{u}_{\tilde{x}}^{(0)} + \mathbf{u}_x^{(1)} + \mathbf{u}_{x^*}^{(2)}) + \ldots$$

and collecting terms according to their powers of ϵ.

3.2.1. $\mathcal{O}(\epsilon^{-1})$ *System*. When we collect all terms that are proportional to ϵ^{-1}, we find that $A(x^*)\ \mathbf{u}_{x^*}^{(0)} = \mathbf{0}$. Because the eigenvalues of $A(x^*)$ are never zero, $A(x^*)$ may be inverted to obtain $\mathbf{u}_{x^*}^{(0)} = \mathbf{0}$. In other words, the leading-order behavior of the solution does not depend on the fast spatial scale: $\mathbf{u}^{(0)} = \underline{\mathbf{u}}^{(0)}(x,\tilde{x},t)$. (As before, we use the convention that all quantities independent of the fast spatial scale are underlined.)

3.2.2. $\mathcal{O}(1)$ *System*. The $\mathcal{O}(1)$ system is

$$\mathbf{u}_{x^*}^{(1)} = -A^{-1}\underline{\mathbf{u}}_t^{(0)} - A^{-1}B\underline{\mathbf{u}}^{(0)} - \underline{\mathbf{u}}_x^{(0)}. \tag{28}$$

Before we integrate, we remove terms that are independent of x^* using the averaging operators (see Appendix A). We obtain the x^*-homogenized equation

$$\underline{\mathbf{u}}_t^{(0)} + \langle A^{-1}\rangle^{-1}\underline{\mathbf{u}}_x^{(0)} + \langle A^{-1}\rangle^{-1}\langle A^{-1}B\rangle\underline{\mathbf{u}}^{(0)} = \mathbf{0}. \tag{29}$$

When we integrate the remaining part of (28) with respect to x^*, we obtain

$$\mathbf{u}^{(1)} = -[\![A^{-1}]\!]\underline{\mathbf{u}}^{(0)} - [\![A^{-1}B]\!]\underline{\mathbf{u}}^{(0)} + \underline{\mathbf{u}}^{(1)}(x,\tilde{x},t,\tilde{t}), \tag{30}$$

where $\underline{\mathbf{u}}^{(1)}$ is the constant of integration. Incidentally, Equation (30) also implies that the $\mathcal{O}(\epsilon)$ initial conditions must be

$$\mathbf{h}^{(1)}(x,x^*) = [\![A^{-1}]\!]\langle A^{-1}\rangle^{-1}\left(\frac{d\underline{\mathbf{h}}^{(0)}}{dx} + \langle A^{-1}B\rangle\underline{\mathbf{h}}^{(0)}(x)\right) - [\![A^{-1}B]\!]\underline{\mathbf{h}}^{(0)} + \underline{\mathbf{h}}^{(1)}(x), \tag{31}$$

where $\underline{\mathbf{h}}^{(1)}(x)$ can be specified arbitrarily, but the fluctuating part of $\mathbf{h}^{(1)}(x, x^*)$ cannot. (See Section 2.2.)

3.2.3. $\mathcal{O}(\epsilon)$ *System*. The $\mathcal{O}(\epsilon)$ system is

$$\mathbf{u}_{x^*}^{(2)} = -\mathbf{u}_x^{(1)} - \underline{\mathbf{u}}_{\tilde{x}}^{(0)} - A^{-1}\mathbf{u}_t^{(1)} - A^{-1}\underline{\mathbf{u}}_{\tilde{t}}^{(0)} - A^{-1}B\mathbf{u}^{(1)} + A^{-1}C(\underline{\mathbf{u}}^{(0)},x^*)\underline{\mathbf{u}}^{(0)} + A^{-1}D(\underline{\mathbf{u}}^{(0)},x^*)\underline{\mathbf{u}}_x^{(0)}. \tag{32}$$

We plug in (30) and remove x^*-independent terms to obtain

$$\begin{aligned}\underline{\mathbf{u}}_t^{(1)} + \langle A^{-1}\rangle^{-1}\underline{\mathbf{u}}_x^{(1)} + \langle A^{-1}\rangle^{-1}\langle A^{-1}B\rangle\underline{\mathbf{u}}^{(1)} &= -\langle A^{-1}\rangle^{-1}\underline{\mathbf{u}}_{\tilde{x}}^{(0)} - \underline{\mathbf{u}}_{\tilde{t}}^{(0)}\\ &+\langle A^{-1}\rangle^{-1}\langle A^{-1}[\![A^{-1}]\!]\rangle\underline{\mathbf{u}}_{tt}^{(0)} + \langle A^{-1}\rangle^{-1}\langle A^{-1}[\![A^{-1}B]\!]\rangle\underline{\mathbf{u}}_t^{(0)}\\ &+\langle A^{-1}\rangle^{-1}\langle A^{-1}B[\![A^{-1}]\!]\rangle\underline{\mathbf{u}}_t^{(0)} + \langle A^{-1}\rangle^{-1}\langle A^{-1}B[\![A^{-1}B]\!]\rangle\underline{\mathbf{u}}^{(0)}\\ &+\langle A^{-1}\rangle^{-1}\langle A^{-1}C(\underline{\mathbf{u}}^{(0)},x^*)\rangle\underline{\mathbf{u}}^{(0)} + \langle A^{-1}\rangle^{-1}\langle A^{-1}D(\underline{\mathbf{u}}^{(0)},x^*)\rangle\underline{\mathbf{u}}_x^{(0)}.\end{aligned} \tag{33}$$

When we integrate the remaining part of (32) with respect to x^*, we obtain

$$\begin{aligned}\mathbf{u}^{(2)} = &[\![[\![A^{-1}]\!]]\!]\underline{\mathbf{u}}_{tx}^{(0)} + [\![[\![A^{-1}B]\!]]\!]\underline{\mathbf{u}}_x^{(0)} + [\![A^{-1}[\![A^{-1}]\!]]\!]\underline{\mathbf{u}}_{tt}^{(0)} + [\![A^{-1}[\![A^{-1}B]\!]]\!]\underline{\mathbf{u}}_t^{(0)}\\ &+ [\![A^{-1}B[\![A^{-1}]\!]]\!]\underline{\mathbf{u}}_t^{(0)} + [\![A^{-1}B[\![A^{-1}B]\!]]\!]\underline{\mathbf{u}}^{(0)} + [\![A^{-1}C(\underline{\mathbf{u}}^{(0)},x^*)]\!]\underline{\mathbf{u}}^{(0)}\\ &+ [\![A^{-1}D(\underline{\mathbf{u}}^{(0)},x^*)]\!]\underline{\mathbf{u}}_x^{(0)} - [\![A^{-1}]\!]\underline{\mathbf{u}}_t^{(1)} - [\![A^{-1}B]\!]\underline{\mathbf{u}}^{(1)} + \underline{\mathbf{u}}^{(2)}(x,\tilde{x},t,\tilde{t}),\end{aligned} \tag{34}$$

where $\underline{\mathbf{u}}^{(2)}$ is the constant of integration. By substituting $x = 0$ into (34), we can obtain the specific form of the $\mathcal{O}(\epsilon^2)$ initial conditions that allows for a t^*-independent solution. We do not display it for brevity.

3.2.4. *Solving the homogenized equations.* We now turn our attention to the homogenized systems of equations obtained from the previous analysis. To solve (29), we diagonalize $\langle \mathrm{A}^{-1}\rangle^{-1}$ by defining $\langle \mathrm{A}^{-1}\rangle^{-1} = P\Lambda P^{-1}$, where $\Lambda = \mathrm{diag}[\lambda_1, \lambda_2]$. We choose $\lambda_1 > 0$ and $\lambda_2 < 0$ without any loss of generality, and define the characteristic independent variables $\xi = x - \lambda_1 t$ and $\eta = x - \lambda_2 t$ along with the characteristic dependent variables $\underline{\mathbf{w}}^{(i)} = P^{-1}\underline{\mathbf{u}}^{(i)}$. We change derivatives using the formulas $\partial_t = -\lambda_1\partial_\xi - \lambda_2\partial_\eta$ and $\partial_x = \partial_\xi + \partial_\eta$. All of these substitutions result in

$$\mathscr{L}^{(\mathrm{h})}[\underline{\mathbf{w}}^{(0)}] \stackrel{\mathrm{def}}{=} \begin{bmatrix} (\lambda_1 - \lambda_2)\underline{w}^{(0)}_{1\eta} \\ (\lambda_2 - \lambda_1)\underline{w}^{(0)}_{2\xi} \end{bmatrix} + P^{-1}\langle A^{-1}\rangle^{-1}\langle A^{-1}B\rangle P\underline{\mathbf{w}}^{(0)} = \mathbf{0}. \tag{35}$$

The differential operator $\mathscr{L}^{(\mathrm{h})}$ governs the x^*-homogenized equations at every order of ϵ. We do not find out how $\underline{\mathbf{u}}^{(0)}$ depends on the slower scales $\tilde{x}$ or $\tilde{t}$ until we consider the set of equations at the next order of ϵ.

Now the reason why we examined $B(x^*) = 0$ case in Section 3.1 becomes apparent—unless one of the off-diagonal terms of $P^{-1}\langle A^{-1}\rangle^{-1}\langle A^{-1}B\rangle P$ vanishes, the equations in (35) are essentially coupled, and their general solution can only be written in terms of integrals of Bessel functions (see Section 3.7.1 of [12]).

The $\mathscr{O}(\epsilon)$ system of x^*-homogenized equations (33), written using the characteristic dependent variables, is

$$\begin{aligned}
\mathscr{L}^{(\mathrm{h})}[\underline{\mathbf{w}}^{(1)}] = {} & -\underline{\mathbf{w}}^{(0)}_{\tilde{x}} - \underline{\mathbf{w}}^{(0)}_{\tilde{t}} + P^{-1}\langle A^{-1}\rangle^{-1}\langle A^{-1}[\![A^{-1}]\!]\rangle P\underline{\mathbf{w}}^{(0)}_{tt} \\
& + P^{-1}\langle A^{-1}\rangle^{-1}\langle A^{-1}[\![A^{-1}B]\!]\rangle P\underline{\mathbf{w}}^{(0)}_{t} \\
& + P^{-1}\langle A^{-1}\rangle^{-1}\langle A^{-1}B[\![A^{-1}]\!]\rangle P\underline{\mathbf{w}}^{(0)}_{t} \\
& + P^{-1}\langle A^{-1}\rangle^{-1}\langle A^{-1}B[\![A^{-1}B]\!]\rangle P\underline{\mathbf{w}}^{(0)} \\
& + P^{-1}\langle A^{-1}\rangle^{-1}\langle A^{-1}C(P\underline{\mathbf{w}}^{(0)}, x^*)\rangle P\underline{\mathbf{w}}^{(0)} \\
& + P^{-1}\langle A^{-1}\rangle^{-1}\langle A^{-1}D(P\underline{\mathbf{w}}^{(0)}, x^*)\rangle P\underline{\mathbf{w}}^{(0)}_{x}.
\end{aligned} \tag{36}$$

Note that (36) is essentially the same as (35), except with source terms. At this stage, we must remove terms from the right-hand side that could potentially lead to secular terms when solving for $\underline{\mathbf{w}}^{(1)}$. In the linear problem of Section 3.1, this task was easy because solving the x^*-homogenized operator only involved integrating the first component with respect to η and the second with respect to ξ. For the general case, we must remove all terms from the right-hand side of (36) that are homogeneous solutions of the homogenized operator $\mathscr{L}^{(\mathrm{h})}$. It is easy to see that $\underline{\mathbf{w}}^{(0)}$, $\underline{\mathbf{w}}^{(0)}_{\tilde{x}}$ and $\underline{\mathbf{w}}^{(0)}_{\tilde{t}}$ are in the null space of $\mathscr{L}^{(\mathrm{h})}$, but it is difficult to see which of the remaining terms needs to be removed.

The perturbation analysis for this class of problems has not been worked out yet.

Let us look at the different possibilities of consistency conditions that can arise. In the most degenerate case when all the matrices on the right-hand side of (36) are zero, the consistency conditions will indicate that $\tilde{x}$ and $\tilde{t}$ should be replaced with $\hat{x} = \epsilon^2 x$ and $\hat{t} = \epsilon^2 t$, and that the analysis must be carried out to $\mathcal{O}(\epsilon^2)$ to see any long-term effects.

The other extreme occurs when every term on the right-hand side of (36) contributes to the consistency condition. In this scenario, the consistency conditions will be a pair of coupled, constant-coefficient, Burgers' equations. The presence of two derivatives with respect to t indicates that diffusion is involved, and that we will not have enough boundary conditions to solve the consistency conditions. (When the consistency conditions are written in physical variables, the terms with two t derivatives become spatial or temporal derivatives depending on whether $\tilde{x}$ or $\tilde{t}$ is involved.) We would have to recover the boundary information using a reduced set of multiple scales (see Section 3.1.7 or Section 4). However, notice that in (36) the matrix $A(x^*)$ is solely responsible for the presence of terms with two t derivatives. In Section 3.1.9, we determined the condition for the diagonal entries of $P^{-1}\langle A^{-1}\rangle^{-1}\langle A^{-1}[\![A^{-1}]\!]\rangle P$ to vanish. We do not know of any physical problems in which this condition is violated.

This observation leads us to believe that the most likely possibility is the one between these two extremes, in which the consistency conditions become a pair of coupled, constant-coefficient, quasilinear, first-order partial differential equations. In this situation, the fact that the consistency condition and the original PDE have the order of derivatives in x and t suggests that missing boundary conditions are not a difficulty. Unfortunately, the absence of terms with two derivatives in t means that there is no diffusion to regulate the onset of shocks. As we discussed in Section 2.2, the moment a shock forms we must include $t^* = t/\epsilon$ in our asymptotic expansion and a general solution will be out of reach.

Let us illustrate these ideas with an example. The governing equations for elastic waves in a one-dimensional solid are

$$\frac{\partial}{\partial t}(\rho(x^*)V) - \frac{\partial}{\partial x}(S(F)) = 0$$

$$\frac{\partial F}{\partial t} - \frac{\partial V}{\partial x} = 0,$$

where $V(x,t;\epsilon)$ is the velocity, $F(x,t;\epsilon)$ is the displacement gradient, $S(F)$ is the stress, and $\rho(x^*)$ is the density, which is allowed to vary on the fast scale, $x^* = x/\epsilon$. (See [13] or [1] for a thorough discussion of elastic waves.) We perturb about the trivial resting state $F = V = 0$ by introducing $V = \epsilon u_1$ and $F = \epsilon u_2$. We also suppose that the relationship between the stress and the displacement gradient is $S(F) = F + \frac{1}{2}F^2$. (We just pick some numbers for simplicity, instead of the more

general $S(F) = \alpha F + \beta F^2$.) The resulting equations for $\mathbf{u}$ can be written in the form of (25) with $B(x^*) = C(\mathbf{u}, x^*) = 0$,

$$A(x^*) = \begin{bmatrix} 0 & -1/p(x^*) \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad D(\mathbf{u}, x^*) = \begin{bmatrix} 0 & u^2/p(x^*) \\ 0 & 0 \end{bmatrix}.$$

The fact that $B(x^*) = 0$ simplifies the analysis of the homogenized equations.

Let us solve an initial-boundary value problem in the domain $x > 0$ and $t > 0$. We will use $\tilde{x}$ but not $\tilde{t}$ in our asymptotic expansion. It is not hard to show that the long-term evolution equations that arise from removing potentially secular terms in (36) are

$$\underline{w}_{1\tilde{x}}^{(0)} + \frac{1}{2}\left(\langle\rho\rangle^{-1/2} - 1\right)\underline{w}_{1\xi}^{(0)}\underline{w}_1^{(0)} = 0 \tag{37a}$$

$$\underline{w}_{2\tilde{x}}^{(0)} - \frac{1}{2}\left(\langle\rho\rangle^{-1/2} + 1\right)\underline{w}_{2\eta}^{(0)}\underline{w}_2^{(0)} = 0. \tag{37b}$$

This pair of decoupled, quasilinear, first-order partial differential equations can be solved without requiring any additional initial and boundary information. To see this, we need to examine the solution domains for both equations. The left half of Figure 6 shows the solution domain for $\underline{w}_1^{(0)}$ in the ξ-$\tilde{x}$ plane and the right half shows the solution domain for $\underline{w}_2^{(0)}$ in the η-$\tilde{x}$ plane.

Initial conditions for $\underline{w}_1^{(0)}$ and $\underline{w}_2^{(0)}$ are specified along the rays $\tilde{x} = \epsilon\xi$ and $\tilde{x} = \epsilon\eta$, respectively; the boundary condition is specified as a linear combination of $\underline{w}_1^{(0)}$ and $\underline{w}_2^{(0)}$ along the rays marked "BC" in Figure 6. As long as there are characteristics filling the solution domain for $\underline{w}_2^{(0)}$, we should be able to calculate $\underline{w}_2^{(0)}$ using the method of characteristics. We can then use the boundary information at $\eta = \tilde{x} = 0$ to obtain a complete solution for $\underline{w}_1^{(0)}$.

What are the ways this method could fail? Because the consistency conditions (37) are homogeneous, every characteristic is a straight line whose slope is determined solely by the value of $\underline{w}_i^{(0)}$ at the point from which the characteristic emanates. Momentarily putting aside the case of shocks, let us

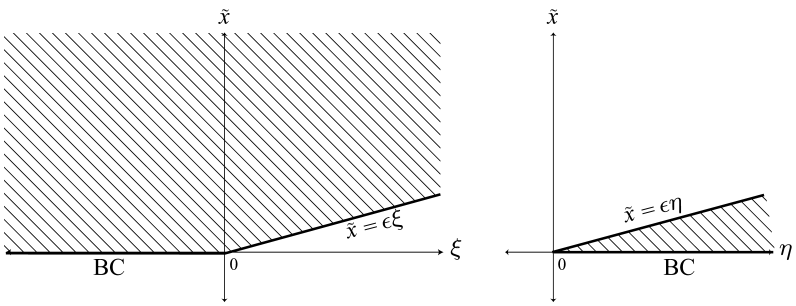


Figure 6. Solution domains for the consistency conditions for elastic waves.

consider the possibility that there are regions of the solution domains that are not covered by characteristics. If we disallow discontinuous initial and boundary data, the only way for the characteristics not to fill the solution domains is if the characteristics have slopes between 0 and ϵ. An example of a "bad" characteristic is shown in Figure 7.

Calculating the slopes of the characteristics, we obtain the following restriction on the initial conditions for the characteristics to fill the solution domain properly:

$$\frac{1}{2}\left(\langle\rho\rangle^{-1/2} - 1\right)\underline{w}_1^{(0)}\Big|_{t=0} < \frac{1}{\epsilon} \tag{38a}$$

and

$$\frac{1}{2}\left(\langle\rho\rangle^{-1/2} + 1\right)\underline{w}_2^{(0)}\Big|_{t=0} > -\frac{1}{\epsilon}. \tag{38b}$$

The initial values of $\underline{\mathbf{w}}^{(0)}$ are obtained from $\underline{\mathbf{u}}^{(0)}$ by $\underline{\mathbf{w}}^{(0)} = P^{-1}\underline{\mathbf{u}}^{(0)}$, where

$$P = \begin{bmatrix} \langle\rho\rangle^{-1/2} & \langle\rho\rangle^{-1/2} \\ -1 & 1 \end{bmatrix}.$$

Note that these two inequalities are implicitly satisfied, because we perturbed about the steady-state $V = F = 0$ to obtain this problem. The initial conditions, therefore, cannot not be on the order of ϵ^{-1}.

The only difficulty remaining is that of shocks. Recall from our discussion in Section 2.2 that the formation of shocks necessitates the introduction of $t^* = t/\epsilon$ into our asymptotic expansion, putting an analytic solution out of reach. To avoid shocks forming from the outset of the problem, we must choose initial conditions that are continuous and that match the boundary

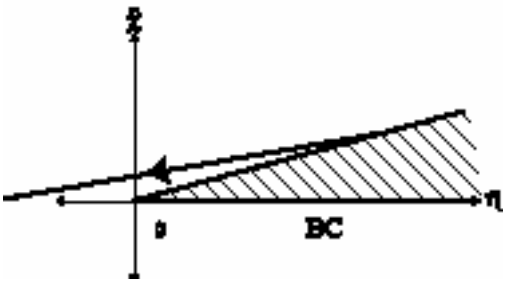


Figure 7. Example of a "bad" characteristic for the second consistency condition for elastic waves.

condition at $x = t = 0$ so that there is a single characteristic emanating from the origin in the ξ-$\tilde{x}$ plane.

As an illustration, let us calculate the solution to this problem with the initial conditions

$$\mathbf{u}(x, 0; \epsilon) = \underline{\mathbf{h}}^{(0)}(x) + \mathcal{O}(\epsilon) = \begin{bmatrix} x/(x+1) \\ 0 \end{bmatrix} + \mathcal{O}(\epsilon)$$

for $x > 0$, and the boundary condition

$$u_1(0, t; \epsilon) = \frac{t + \frac{1}{2}\sin t}{t+3} + \mathcal{O}(\epsilon)$$

for $t > 0$. We choose $\epsilon = 0.1$ and $\rho(x^*) = 1 + \frac{1}{2}\sin x^*$ so $\langle\rho\rangle = 1$. The initial conditions satisfy (38) and match the boundary condition at $x = t = 0$. The characteristic emanating from the origin is the line $x = \langle\rho\rangle^{-1/2}t$. Figure 8 shows the analytic solution at two different times. The nonlinear effects, though difficult to observe from these graphs, cause slight wave-steepening in the solution. The reason that the solutions for u_1 and u_2 appear symmetric is because $\underline{h}_2^{(0)}(x) = 0$.

3.2.5. *Summary and discussion.* We have seen that all of the basic concepts from the linear problem with $B(x^*) = 0$ apply equally well to the general nonlinear problem (25). These concepts include the relationship between slow and stretched scales and observable long-term phenomena, the construction of consistency conditions, the reason for the missing boundary difficulty, and the recovery of missing boundary information using a reduced set of multiple scales. Generalizing to (25), however, does alter the analysis in a few ways.

First, if $B(x^*)$ is nonzero, the operator $\mathcal{L}^{(h)}$ that governs the x^*-homogenized equations to every order of ϵ is likely to represent an essentially coupled system of equations. (More precisely, this happens when $P^{-1}\langle A^{-1}\rangle^{-1}\langle A^{-1}\ B\rangle\ P$ has

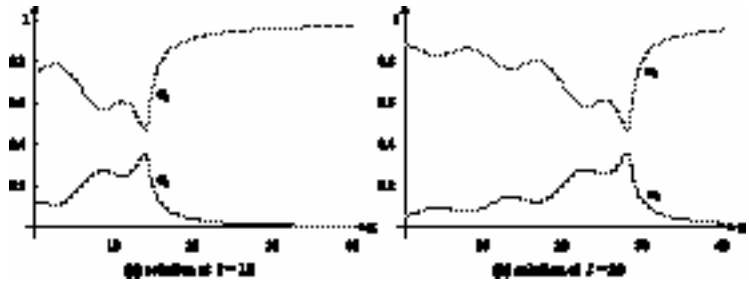


Figure 8. Analytic solution to the long-term evolution equations (consistency conditions) for elastic waves at two different times.

nonzero off-diagonal terms.) The result of this coupling is that the general solution of the homogenized equations involves integrals of Bessel functions, and more importantly, the construction of the consistency conditions from (36) is obscured. The consistency conditions have the potential to form a pair of coupled Burgers' equations, whose solution necessitates the recovery of boundary information at $x = 0$. In Section 4, we show how to recover the boundary information for a linear problem with nontrivial $B(x^*)$—this is sufficient to recover the leading-order boundary information for the general nonlinear problem.

Second, the fact that (25) is nonlinear means that we cannot use superposition to partition an initial-boundary value problem into an initial-value problem and a signaling problem. Although it is not any more difficult to recover boundary information for a problem with nontrivial rather than trivial initial conditions, we have seen in Section 3.1.7 that nontrivial initial conditions lead to secular terms in the recovered boundary information. These secular terms limit the region of validity of the computed solution.

Finally, the presence of nonlinearities can actually alleviate the missing boundary difficulty. Whether diffusion is present in the consistency condition (36) depends only on the matrix $A(x^*)$, and we have not come across any physically relevant problems that exhibit diffusion. From our analysis of the linear problem with $B(x^*) = 0$, we learned that after diffusion, the next possible type of long-term behavior is dispersion, which appears in the $\mathcal{O}(\epsilon^2)$ system of homogenized equations. However, the most significant long-term effects for nonlinear problems already manifest themselves at the $\mathcal{O}(\epsilon)$ system of x^*-homogenized equations. Because these nonlinearities occur one order of ϵ earlier, and only involve one spatial derivative of the dependent variables, we predict that the consistency conditions to leading order for all physically relevant problems will be first-order partial differential equations. Because the order of these consistency conditions matches the original partial differential equation, no additional boundary conditions are needed to solve the consistency conditions. The initial-boundary value problem of elastic waves in a one-dimensional solid is a perfect example of such a situation.

4. The Laplace-multiple-scales method

In this section, we present an alternative method for handling linear hyperbolic systems with rapid spatial fluctuations using a combination of Laplace transforms and multiple-scales (the Laplace-MS method). In the interest of space, we only outline the method and point out its usefulness. Full details can be found in [9].

The main idea is to use a Laplace transform in time to turn PDEs into ODEs, then apply the multiple-scales method to the resulting ODEs. We only apply these methods to problems fitting equation (2), because the Laplace transform is only suited for linear problems. Applying multiple-scales to the resulting ODEs is rather straightforward. The unknown functions are expanded with the same set of spatial scales used for the usual multiple-scale procedure without Laplace transforms.

In [9], we show that this Laplace-MS method is essentially the same as the usual multiple-scale method. The missing boundary condition difficulty manifests itself in the Laplace-MS method through noninvertible terms like $\exp(s^n)$, where n is the order of the consistency condition that causes the difficulty in the usual multiple-scale method. Furthermore, the Laplace-MS method can be used to recover missing boundary information by solving the problem with a reduced set of scales. All of the results of the previous section have been verified using the Laplace-MS method: the missing boundary conditions for the linear wave equation (with trivial and nontrivial initial conditions), and the observation that secular terms appear in the recovered boundary information if the initial conditions are nontrivial.

However, the reason we bring up this Laplace-MS method is that it requires much less algebra to recover missing boundary information, especially when working with PDEs of the form (2) with $B \neq 0$. We end this section by demonstrating the recovery of boundary information for such a problem.

Consider Maxwell's equations specialized for plane polarized waves, propagating in a one-dimensional medium. If the current density is linearly related to the electric field, the governing equations may be expressed in the form of (2), where

$$A(x^*) = \begin{bmatrix} 0 & \varepsilon(x^*)^{-1} \\ \mu(x^*)^{-1} & 0 \end{bmatrix} \quad \text{and} \quad B(x^*) = \begin{bmatrix} \sigma(x^*) & 0 \\ 0 & 0 \end{bmatrix}.$$

Here, $\varepsilon(x^*)$ (different from ϵ), $\mu(x^*)$, and $\sigma(x^*)$ are the dielectric constant, permeability, and conductivity, respectively. The first dependent variable, $u_1(x,t;\epsilon)$, is the electric field, and the second is the magnetic field. All variables have been appropriately nondimensionalized. (These equations are derived in [1] for a nonlinear medium.)

Suppose we have a signaling problem in which $u_1(0, t; \epsilon) = g(t; \epsilon) = g^{(0)}(t) + \mathcal{O}(\epsilon)$ and $\underline{\mathbf{u}}^{(0)}(x, 0) = \mathbf{0}$. The matrix $Z = A^{-1}(sI + B)$ has the average value

$$\langle Z \rangle = \begin{bmatrix} 0 & \langle \mu \rangle s \\ \langle \varepsilon \rangle s + \langle \varepsilon \sigma \rangle & 0 \end{bmatrix}.$$

Using the Laplace-MS method for recovering boundary conditions, we find that the recovered boundary information is

$$\underline{u}_2^{(0)}(0,t) = \alpha\sqrt{\frac{\langle\varepsilon\rangle}{\langle\mu\rangle}}\int_0^t e^{-\alpha\tau}[\mathscr{I}_0(\alpha\tau) + \mathscr{I}_1(\alpha\tau)]g^{(0)}(t-\tau)d\tau,$$

where $\mathscr{I}_n(x)$ is the modified Bessel's function of the first kind, and $\alpha = 2\langle\varepsilon\sigma\rangle/\langle\varepsilon\rangle$. Performing this calculation using the usual multiple-scale method requires much more algebra.

5. Summary and conclusion

We began this article by asking what role boundary conditions play in the multiple-scale analysis of hyperbolic conservation laws with rapidly fluctuating coefficients. The answer is that boundary conditions introduce a serious mathematical difficulty (the missing boundary condition difficulty), which arises because the homogenized equations are typically of higher order than the original problem. We have overcome this difficulty through the recovery of missing boundary information. Using either the usual multiple-scale method or the Laplace-MS method, the recovery procedure is the same.

1. Use a reduced set of scales to solve the original problem, thereby eliminating the need to solve long-term evolution equations (consistency conditions).
2. Generate a near-field solution, ignoring the fact that it has a relatively small region of validity.
3. Extract the necessary boundary information to solve the long-term evolution equations.

We applied this procedure to both linear and nonlinear problems in Section 3, using the usual multiple-scale method.

The Laplace-MS method is essentially the same as the usual multiple-scale method except that it only applies to linear problems, and it can recover boundary information with much less algebra for problems in which $B(x^*) \neq 0$. The fact that the Laplace-MS method gives the same results as the usual multiple-scale method is expected because we can usually interchange derivatives and integrals with respect to x and t.

We have seen that the recovered boundary information will contain secular terms in t when the initial conditions are nontrivial. These secular terms limit the region of validity of the recovered boundary information. Fortunately, for linear problems, we can overcome this difficulty with the aid of superposition. By partitioning an initial-boundary value problem into the sum of an initial-value problem and a signaling problem, we can use the recovered boundary

information for a signaling problem, which will not contain secular terms, to solve the long-term evolution equations.

The multiple-scale analysis for problem (1) is no more conceptually difficult, although there are a few procedural complications. First, the linear operator that governs the x^*-homogenized equations at every order of ϵ is likely to represent a pair of essentially coupled equations, when $B(x^*)$ is nonzero. This coupling causes the general solution to involve Bessel functions, and it obscures the construction of consistency conditions. Although the asymptotic solution of the x^*-homogenized equation is generally out of reach for $B \neq 0$, these equations provide a useful description of how the x^*-averaged solution behaves, and may be solved numerically with a much coarser grid than as the x^*-dependent terms are absent.

Second, the presence of nonlinear terms in (1) means that an initial-boundary value problem cannot be decomposed into an initial-value problem and a signaling problem using superposition. The recovered boundary information for a problem with nontrivial initial conditions is likely to contain secular terms proportional to t, which limit the region of validity of the asymptotic expansion. However, because the appearance of diffusion is governed solely by the linear part of the problem, and to this date, we have not encountered a physically relevant problem that includes diffusion, we conjecture that the consistency conditions (long-term evolution equations) for physically relevant nonlinear problems will not include diffusion. Instead, they will form a pair of first-order, quasilinear PDEs. Because the order of these consistency conditions matches the original partial differential equation, no additional boundary conditions are needed to solve the consistency conditions.

Appendix A. Averaging operators

To handle functions that vary on the fast spatial scale $x^* = x/\epsilon$, we introduce the following operators.

$$\langle a(x^*)\rangle \stackrel{\text{def}}{=} \lim_{s\to\infty} \frac{1}{2s} \int_{-s}^{s} a(s)ds \qquad \{a(x^*)\} \stackrel{\text{def}}{=} a(x^*) - \langle a(x^*)\rangle$$

$$[\![a(x^*)]\!] \stackrel{\text{def}}{=} \int_{s}^{x^*} \{a(\xi)\}d\xi \text{ where } s \text{ is chosen such that } \langle\{a(x^*)\}\rangle = 0.$$

The averaging operator $\langle\cdot\rangle$ gives the average value of a function over the entire x^*-domain. It is a linear operator that produces constants or functions that are

independent of x^*. For a function $b(x^*)$ with period $2P$, the averaging operator is more conveniently defined as

$$\langle b(x^*)\rangle \stackrel{\text{def}}{=} \frac{1}{2P}\int_{-P}^{P} b(s)ds.$$

The $\{\cdot\}$ operator generates the fluctuating part of a function: the part of the function that has a zero average. Finally, the $[\![\cdot]\!]$ operator gives the integral of the fluctuating part of a function, where the constant of integration is chosen such that the average of the integral of the fluctuating part is zero. (Keep in mind that $\{a(x^*)\}$ is still a function of x^*, whereas $\langle a(x^*)\rangle$ is not.) When any of these operators is applied to a matrix, it is understood that the operation is performed element-wise.

We now present some interesting and useful properties of these averaging operators. In the following discussion, it is understood that $a = a(x^*)$ and $b = b(x^*)$ are integrable, although not necessarily periodic or differentiable with respect to x^*.

Property A.1.

$$\frac{d}{dx^*}[\![a]\!] = \{a\}.$$

This proposition is a direct consequence of the fact that $[\![\cdot]\!]$ is an integral operator. The function $a(x^*)$ does not have to be differentiable, only integrable.

Property A.2.

$$\left[\!\left[\frac{d}{dx^*}a(x^*)\right]\!\right] = \frac{d}{dx^*}[\![a(x^*)]\!] \qquad \text{for all differentiable } a(x^*).$$

To prove this property, we use the definition of the $[\![\cdot]\!]$ operator:

$$\left[\!\left[\frac{d}{dx^*}a(x^*)\right]\!\right] = \int_s^{x^*} a'(\xi)d\xi = a(x^*) - a(s),$$

where s is chosen such that

$$\left\langle \left[\!\left[\frac{d}{dx^*}a(x^*)\right]\!\right] \right\rangle = \langle a(x^*) - a(s)\rangle = \langle a(x^*)\rangle - a(s) = 0.$$

Substituting $a(s) = \langle a(x^*)\rangle$,

$$\left[\!\left[\frac{d}{dx^*}a(x^*)\right]\!\right] = a(x^*) - \langle a(x^*)\rangle = \{a(x^*)\} = \frac{d}{dx^*}[\![a]\!].$$

In effect, this property allows us to interchange differentiation and integration. It is the only property listed here that requires $a(x^*)$ to be differentiable.

Property A.3.

$$[\![a]\!][\![b]\!] = [\![[\![a]\!]\{b\} + \{a\}[\![b]\!]]\!] + \langle [\![a]\!][\![b]\!] \rangle.$$

Proof:

$$[\![a]\!][\![b]\!] = \{[\![a]\!][\![b]\!]\} + \langle [\![a]\!][\![b]\!] \rangle$$

$$\{[\![a]\!][\![b]\!]\} = \frac{d}{dx^*}[\![[\![a]\!][\![b]\!]]\!] \quad \text{Use Property A.1.}$$

$$\text{Use Property A.2.} = \left[\!\!\left[\frac{d}{dx^*}([\![a]\!][\![b]\!]) \right]\!\!\right]$$

$$\text{Use Property A.1.} = [\![[\![a]\!]\{b\} + \{a\}[\![b]\!]]\!].$$

Note: Although this property makes use of Property A.2, it does not require $a(x^*)$ or $b(x^*)$ to be differential, because if these functions are integrable, then $[\![a]\!]\ [\![b]\!]$ is differentiable.

Property A.4.

$$\langle [\![a]\!]a \rangle = 0 \text{ for all integrable, scalar functions } a(x^*).$$

Proof:

$$\langle [\![a]\!]a \rangle = \langle [\![a]\!](\langle a \rangle + \{a\}) \rangle = \langle a \rangle \langle [\![a]\!] \rangle + \langle [\![a]\!]\{a\} \rangle = \langle [\![a]\!]\{a\} \rangle$$

$$\text{Use Property A.1.} = \left\langle \frac{1}{2} \frac{d}{dx^*} \left([\![a]\!]^2 \right) \right\rangle$$

$$\text{Use Property A.3.} = \frac{1}{2} \left\langle \frac{d}{dx^*} \left(2[\![[\![a]\!]\{a\}]\!] + \left\langle [\![a]\!]^2 \right\rangle \right) \right\rangle$$

$$= \langle \{[\![a]\!]\{a\}\} \rangle = 0$$

Property A.5.

$$\langle a[\![b]\!] \rangle = -\langle [\![a]\!]b \rangle$$

We prove this statement by applying Property A.4 to the equality

$$\langle [\![a+b]\!](a+b) \rangle = \langle [\![a]\!]a \rangle + \langle [\![a]\!]b \rangle + \langle [\![b]\!]a \rangle + \langle [\![b]\!]b \rangle.$$

Because the averaging operator involves an integral, this property is really a manifestation of integration by parts. However, we can define the averaging

operator without mention of integrals, and the result will still be true. (Simply define $\langle\cdot\rangle$ as a linear operator such that $a(x^*) = \langle a\rangle + \{a\}$ and $\langle\alpha\rangle = \alpha$ for all constants α.)

It is very important to remember that Property A.4 does not hold for matrices, because its proof relies on the commutative property of scalar multiplication. The expression $\langle A[\![A]\!]\rangle$ appears many times throughout this article, and this expression is not zero if $A(x^*)$ is a matrix. The most that we can say about $\langle A[\![A]\!]\rangle$ is that if $A(x^*)$ is a square, integrable matrix, then the trace of $\langle A[\![A]\!]\rangle$ is zero. (We can prove this quickly using some linear algebra and Properties A.4 and A.5.)

However, Property A.5 does hold for matrices. For any square matrices $A(x^*)$ and $B(x^*)$ of the same size,

$$\langle A[\![B]\!]\rangle = -\langle [\![A]\!]B\rangle.$$

In particular, if $A = B$, we see that

$$\langle A[\![A]\!]\rangle = -\langle [\![A]\!]A\rangle.$$

Property A.6.

$$\langle a[\![[\![b]\!]]\!]\rangle = -\langle [\![a]\!][\![b]\!]\rangle = \langle [\![[\![a]\!]]\!]b\rangle \text{ for all integrable } a(x^*) \text{ and } b(x^*).$$

This final result is proved through repeated applications of Property A.5.

Appendix B. Numerical methods

In this appendix, we give further details about the two numerical methods used throughout this article: a spectral solver and a finite volume method.

B.1. CLAWPACK

CLAWPACK (Conservation LAWs PACKage), written by Randall J. LeVeque, is a package of Fortran routines that numerically solves hyperbolic systems of conservation laws. (See [14] and [15].) Because it is a finite volume method, it can be used to calculate solutions that truly conserve the appropriate quantities. The fundamental unit of CLAWPACK is a user-supplied routine that solves a Riemann problem—an initial-value problem with piecewise constant data. CLAWPACK handles everything else: time-stepping, flux corrections using flux limiters, even adaptive mesh refinement (with AMRCLAW). It is available on the web at http://www.amath.washington.edu/~rjl/clawpack.html.

B.2. Spectral Solver

The other numerical method that we have used in this article is loosely based on the ideas behind spectral methods. Essentially, we solve the system of ordinary differential equations that results when we assume the solution is a truncated Fourier series.

Consider the system of equations

$$\mathbf{u}_t + A(x/\epsilon)\mathbf{u}_x = 0, \tag{B.1}$$

with periodic $A(x/\epsilon)$ on the domain $-1 < x < 1$ with periodic boundary conditions. If the initial conditions $\mathbf{u}(x, 0)$ are periodic, then the solution will also be periodic for all $t > 0$. We can, therefore, write a Fourier series expansion for the solution:

$$\mathbf{u}(x,t) = \sum_{n=-\infty}^{\infty} \mathbf{v}^{(n)}(t)\cos(n\pi x) + \mathbf{w}^{(n)}(t)\sin(n\pi x). \tag{B.2}$$

Plugging (B.2) into (B.1), we see that the amplitudes $\mathbf{v}^{(n)}(t)$ and $\mathbf{w}^{(n)}(t)$ are governed by an infinite system of ordinary differential equations.

Equation (B.2) suggests a simple approximation to $\mathbf{u}$: truncate the Fourier expansion by defining

$$\tilde{\mathbf{u}}(x,t) = \sum_{n=m}^{M} \mathbf{v}^{(n)}(t)\cos(n\pi x) + \mathbf{w}^{(n)}(t)\sin(n\pi x). \tag{B.3}$$

Now, instead of an infinite system of ordinary differential equations, we obtain a system of $2(M - m + 1)$ differential equations governing the amplitudes. The initial conditions for these ODEs come from a similar truncated Fourier expansion of the initial conditions to the PDE. The system of ODEs is linear and has constant coefficients, so it can be easily solved.

We implemented this numerical method using *Mathematica*. Although *Mathematica* can perform symbolic calculations, in this case, the required matrix exponentiation is too computationally intensive for large systems of ODEs. Instead, we have used *Mathematica*'s arbitrary precision arithmetic to calculate solutions for large t.

We expect that, as more terms are incorporated into the truncated Fourier expansion, the approximation becomes more accurate. However, how many Fourier modes should we include? As an illustration, we solved (B.1) with

$$A_{\text{test}}^{-1}(x^*) = \begin{bmatrix} 1 + 0.5\sin(\pi x^*) & 2 \\ 2 & 1 + 0.25\sin(\pi x^*) \end{bmatrix}$$

with the initial condition

$$\mathbf{u}(x,0) = \begin{bmatrix} \sin(\pi x) \\ 0 \end{bmatrix},$$

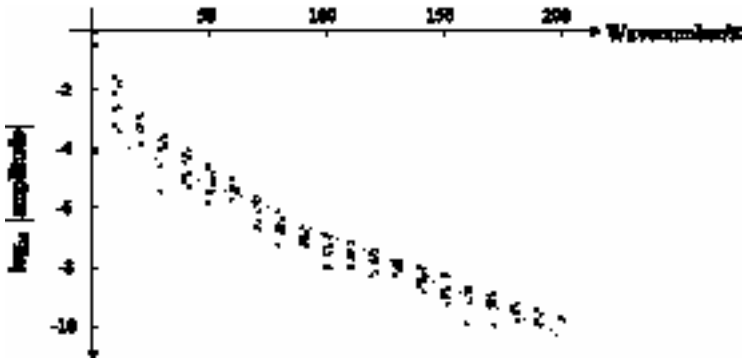


Figure B.1. Amplitudes of Fourier modes present in a spectral solution to the linear wave equation at $t = 500$.

using $m = -200$ and $M = 200$ in the truncated Fourier series (B.3). We used 64-digit arithmetic. Figure B.1 shows the amplitudes of each mode on a logarithmic scale, when $t = 500$. There are four dots for each wavenumber, because we plot the amplitudes of the cosine and sine for both $\tilde{u}_1(x,t)$ and $\tilde{u}_2(x,t)$. The appearance of bands in the graph is a result of our monochromatic initial condition and $A(x/\epsilon)$—with $\epsilon = 0.1$, the initial wave with wavenumber π excites only the waves with wavenumbers $\ldots$, -19π, -9π, π, 11π, 21π,....

The most important feature of Figure B.1 is that the amplitudes decay exponentially with increasing wavenumber. If we continue the trend, we see that the magnitude of the first neglected Fourier mode is approximately 10^{-10}. If we are satisfied with a solution accurate to 10^{-5}, we only need to set $m = -50$ and $M = 50$ in our truncated Fourier series.

We have also used this numeric method to solve the linear wave equation. In general, spectral numeric techniques are typically much more accurate than their finite difference counterparts. However, they are limited to problems with periodic boundary conditions.

Acknowledgments

The authors thank Dr. Randall J. LeVeque, Tiernan Fogarty, and James Rossmanith for their help with CLAWPACK.

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HARVEY MUDD COLLEGE
UNIVERSITY OF WASHINGTON

(Received November 30, 2000)