Harvey Mudd College Math Tutorial:

Antiderivatives

Let $f(x)$ be continuous on $[a,b]$. If $G(x)$ is continuous on $[a,b]$ and $G'(x) = f(x)$ for all $x \in (a,b)$, then G is called an **antiderivative** of f.

We can construct antiderivatives by integrating. The function

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative for f since it can be shown that $F(x)$ constructed in this way is continuous on $[a,b]$ and $F'(x) = f(x)$ for all $x \in (a,b)$.

Properties

Let $F(x)$ be any antiderivative for $f(x)$.

- For any constant C, $F(x) + C$ is an antiderivative for $f(x)$.

  Proof:

  Since $\dfrac{d}{dx}[F(x)] = f(x)$,

  $$\begin{aligned} \frac{d}{dx}[F(x)+C] &= \frac{d}{dx}[F(x)] + \frac{d}{dx}[C] \\ &= f(x) + 0 \\ &= f(x) \end{aligned}$$

  so $F(x) + C$ is an antiderivative for $f(x)$.

- Every antiderivative of $f(x)$ can be written in the form

  $$F(x) + C$$

  for some C. That is, every two antiderivatives of f differ by at most a constant.

  Proof:

  Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$. Then $F'(x) = G'(x) = f(x)$, so $F(x)$ and $G(x)$ differ by at most a constant (this requires proof—it is shown in most calculus texts and is a consequence of the Mean Value Theorem).

The process of finding antiderivatives is called **antidifferentiation** or **integration**:

$$\frac{d}{dx}[F(x)] = f(x) \qquad \Longleftrightarrow \qquad \int f(x)\,dx = F(x) + C.$$

$$\frac{d}{dx}[g(x)] = g'(x) \qquad \Longleftrightarrow \qquad \int g'(x)\,dx = g(x) + C.$$

Properties of the Indefinite Integral

- $\dfrac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$.

  Proof:

  Let $\int f(x)\,dx = F(x)$, where $F(x)$ is an antiderivative of f. Then

$$\begin{aligned} \frac{d}{dx}\left[\int f(x)\,dx\right] &= \frac{d}{dx}F(x) \\ &= f(x). \end{aligned}$$

- *(Linearity)* $\int [\alpha f(x) + \beta g(x)]\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$.

  Proof:

  We need only show that $\alpha \int f(x)\,dx + \beta \int g(x)\,dx$ is an antiderivative of $\int [\alpha f(x) + \beta g(x)]\,dx$:

$$\begin{aligned} \frac{d}{dx}\left[\alpha \int f(x)\,dx + \beta \int g(x)\,dx\right] &= \alpha \frac{d}{dx}\left[\int f(x)\,dx\right] + \beta \frac{d}{dx}\left[\int g(x)\,dx\right] \\ &= \alpha f(x) + \beta g(x). \end{aligned}$$

Examples

1. Every antiderivative of x^2 has the form $\dfrac{x^3}{3} + C$, since $\dfrac{d}{dx}\left[\dfrac{x^3}{3}\right] = x^2$.

2. $\dfrac{d}{dx}\left[\int x^5\,dx\right] = x^5$.

Key Concepts

If $G(x)$ is continuous on $[a, b]$ and $G'(x) = f(x)$ for all $x \in (a, b)$, then G is called an antiderivative of f.

We can construct antiderivatives by integrating. The function $F(x) = \int_a^x f(t)\,dt$ is an antiderivative for f. In fact, every antiderivative of $f(x)$ can be written in the form $F(x) + C$, for some C.

$$\frac{d}{dx}[F(x)] = f(x) \quad \Longleftrightarrow \quad \int f(x)\,dx = F(x) + C.$$

$$\frac{d}{dx}[g(x)] = g'(x) \quad \Longleftrightarrow \quad \int g'(x)\,dx = g(x) + C.$$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]