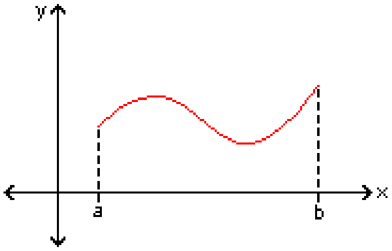
Harvey Mudd College Math Tutorial:

Arc Length

Suppose f is continuously differentiable on the interval $[a, b]$.

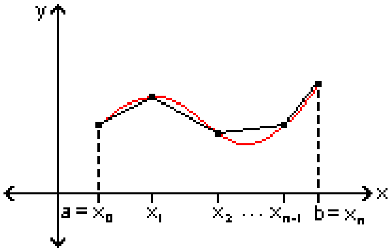
Let's derive a formula for the length L of the curve on the interval, called the *arc length* over $[a, b]$.



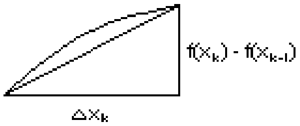
We'll start by subdividing the interval $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$ where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

Introduce the line segments between $(x_0, f(x_0))$ and $(x_1, f(x_1)), (x_1, f(x_1))$ and $(x_2, f(x_2)), \ldots, (x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$.

The resulting polygonal path approximates the curve given by $y = f(x)$, and its length approximates the arc length of $f(x)$ over $[a, b]$.



Let's find the length of the polygonal path by adding up the lengths of the individual line segments. The kth line segment is the hypotenuse of a triangle with base Δx_k and height $f(x_k) - f(x_{k-1})$, and so has length



$$L_k = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}.$$

By the **Mean Value Theorem**, there exists $x_k^* \in [x_{k-1}, x_k]$ such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*)$$

so

$$f(x_k) - f(x_{k-1}) = f'(x_k^*)(x_k - x_{k-1}) = f'(x_k^*)\Delta x_k.$$

Thus,

$$L_k = \sqrt{(\Delta x_k)^2 + [f'(x_k^*)\Delta x_k]^2} = \sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k.$$

Finally, the length of the entire polygonal path is

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k$$

which has the form of a **Riemann sum**. Increasing the number of subintervals such that $\max \Delta x_k \to 0$, $\sum_{k=1}^{n} L_k \to L$. That is,

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

by the definition of the **definite integral** as a limit of Riemann sums. Thus, we have proved the following:

Arc Length

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$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Similarly, if $x = g(y)$ with g continuously differentiable on $[c, d]$, then the arc length L of $g(y)$ over $[c, d]$ is given by

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

These integrals often can only be computed using numerical methods.

Example

We can compute the arc length of the graph of $f(x) = x^{3/2}$ over $[0, 1]$ as follows:

$$\begin{aligned} L = \int_0^1 \sqrt{1 + [f'(x)]^2}\, dx &= \int_0^1 \sqrt{1 + [3x^{1/2}/2]^2}\, dx \\ &= \int_0^1 \sqrt{1 + 9x/4}\, dx \\ &= \frac{8}{27}(1 + 9x/4)^{3/2}\Big|_0^1 \\ &= (1 + 9/4)^{3/2} - (1)^{3/2} \\ &= (13/4)^{3/2} - 1 \\ &\approx 1.44. \end{aligned}$$

Exploration

Key Concepts

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Similarly, if $x = g(y)$ with g continuously differentiable on $[c, d]$, then the arc length L of $g(y)$ over $[c, d]$ is given by

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]