Harvey Mudd College Math Tutorial:

The Binomial Theorem

We know that

$$\begin{aligned}(x+y)^0 &= 1\\(x+y)^1 &= x+y\\(x+y)^2 &= x^2+2xy+y^2\end{aligned}$$

and we can easily expand

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3.$$

For higher powers, the expansion gets very tedious by hand! Fortunately, the Binomial Theorem gives us the expansion for any positive integer power of $(x+y)$:

Binomial Theorem

For any positive integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k$$

where

$$\binom{n}{k} = \frac{(n)(n-1)(n-2)\cdots(n-(k-1))}{k!} = \frac{n!}{k!(n-k)!}.$$

Proof by Induction Combinatorial Induction Connection to Pascal's Triangle

Example

By the Binomial Theorem,

$$\begin{aligned}(x+y)^3 &= \sum_{k=0}^{3}\binom{3}{k}x^{3-k}y^k\\ &= \binom{3}{0}x^3+\binom{3}{1}x^2y+\binom{3}{2}xy^2+\binom{3}{3}y^3\\ &= x^3+3x^2y+3xy^2+y^3\end{aligned}$$

as expected.

Extensions of the Binomial Theorem

A useful special case of the Binomial Theorem is

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

for any positive integer n, which is just the Taylor series for $(1+x)^n$.

This formula can be extended to all real powers α:

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

for any real number α, where

$$\binom{\alpha}{k} = \frac{(\alpha)(\alpha-1)(\alpha-2)\cdots(\alpha-(k-1))}{k!} = \frac{\alpha!}{k!(\alpha-k)!}.$$

Notice that the formula now gives an infinite series. (When $\alpha = n$ is a positive integer, all but the first $(n+1)$ terms are 0 since after this $n-n$ $(=0)$ appears in each numerator.)

This expansion is very useful for approximating $(1+x)^\alpha$ for $|x| \ll 1$:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots.$$

But for $|x| \ll 1$, higher powers of x get small very quickly, so $(1+x)^\alpha$ can be approximated to any accuracy we need by truncating the series after a finite number of terms.

Example
For $|x| \ll 1$,

$$\begin{aligned}(1+x)^{5/2} &\approx 1 + \frac{5}{2}x,\\ (1-2x)^{100} &\approx 1 - 200x,\\ (1+x^2)^{-3} &\approx 1 - 3x^2.\end{aligned}$$

This type of reasoning is useful in investigating what happens when a physical system is perturbed slightly, introducing a new very small term x.

Key Concepts

Binomial Theorem

For any positive integer n,

$$(x+y)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^{n-k} y^k$$

where

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]