

Harvey Mudd College Math Tutorial:

The Chain Rule

You probably remember the derivatives of $\sin(x)$, x^8 , and e^x . But what about functions like $\sin(2x - 1)$, $(3x^2 - 4x + 1)^8$, or e^{-x^2} ? How do we take the derivative of **compositions** of functions?

The **Chain Rule** allows us to use our knowledge of the derivatives of functions $f(x)$ and $g(x)$ to find the derivative of the composition $f(g(x))$:

Suppose a function $g(x)$ is differentiable at x and $f(x)$ is differentiable at $g(x)$. Then the composition $f(g(x))$ is differentiable at x .

Letting $y = f(g(x))$ and $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Using alternative notation,

$$\begin{aligned}\frac{d}{dx} [f(g(x))] &= f'(g(x))g'(x), \\ \frac{d}{dx} [f(u)] &= f'(u) \frac{du}{dx}.\end{aligned}$$

Proof

The three formulations of the Chain Rule given here are identical in meaning. In words, the derivative of $f(g(x))$ is the derivative of f , evaluated at $g(x)$, multiplied by the derivative of $g(x)$.

Examples

- To differentiate $\sin(2x - 1)$, we identify $u = 2x - 1$. Then

$$\begin{aligned}\frac{d}{dx} [\sin(2x - 1)] &= \frac{d}{dx} [\sin(u)] \cdot \frac{d}{dx} [2x - 1] \\ &= \cos(u) \cdot 2 \\ &= 2 \cos(2x - 1).\end{aligned}$$

$$\begin{aligned}f(x) &= \sin(x) \\ g(x) &= 2x - 1 \\ f(g(x)) &= \sin(2x - 1)\end{aligned}$$

- To differentiate $(3x^2 - 4x + 1)^8$, we identify $u = 3x^2 - 4x + 1$. Then

$$\begin{aligned} \frac{d}{dx} \left[(3x^2 - 4x + 1)^8 \right] &= \frac{d}{du} [u^8] \cdot \frac{d}{dx} [3x^2 - 4x + 1] \\ &= 8u^7 \cdot (6x - 4) \\ &= 8(6x - 4) (3x^2 - 4x + 1)^7. \end{aligned}$$

$$\begin{aligned} f(x) &= x^8 \\ g(x) &= 3x^2 - 4x + 1 \\ f(g(x)) &= (3x^2 - 4x + 1)^8 \end{aligned}$$

- To differentiate e^{-x^2} , we identify $u = -x^2$. Then

$$\begin{aligned} \frac{d}{dx} [e^{-x^2}] &= \frac{d}{du} [e^u] \cdot \frac{d}{dx} [-x^2] \\ &= e^u \cdot (-2x) \\ &= -2xe^{-x^2}. \end{aligned}$$

$$\begin{aligned} f(x) &= e^x \\ g(x) &= -x^2 \\ f(g(x)) &= e^{-x^2} \end{aligned}$$

Sometimes you will need to apply the Chain Rule several times in order to differentiate a function.

Example

We will differentiate $\sqrt{\sin^2(3x) + x}$.

$$\begin{aligned} \frac{d}{dx} \left[\sqrt{\sin^2(3x) + x} \right] &= \frac{1}{2\sqrt{\sin^2(3x) + x}} \cdot \frac{d}{dx} [\sin^2(3x) + x] & f(u) &= \sqrt{u} \\ &= \frac{1}{2\sqrt{\sin^2(3x) + x}} \cdot \left(2 \sin(3x) \frac{d}{dx} [\sin(3x)] + 1 \right) & f(u) &= u^2 \\ & & \frac{d}{dx} [x] &= 1 \\ &= \frac{1}{2\sqrt{\sin^2(3x) + x}} \cdot \left(2 \sin(3x) \cos(3x) \frac{d}{dx} [3x] + 1 \right) & f(u) &= \sin(u) \\ &= \frac{1}{2\sqrt{\sin^2(3x) + x}} \cdot (2 \sin(3x) \cos(3x) \cdot 3 + 1) \\ &= \frac{6 \sin(3x) \cos(3x) + 1}{2\sqrt{\sin^2(3x) + x}} \end{aligned}$$

Key Concepts

Let $g(x)$ be differentiable at x and $f(x)$ be differentiable at $f(g(x))$. Then, if $y = f(g(x))$ and $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

[I'm ready to take the quiz.] [I need to review more.]
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