Harvey Mudd College Math Tutorial:

Change of Basis

Let V be a vector space and let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a set of vectors in V. Recall that S forms a **basis** for V if the following two conditions hold:

1. S is **linearly independent**.
2. S **spans** V.

If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for V, then every vector $\mathbf{v} \in V$ can be expressed *uniquely* as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$:

$$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}.$$

Think of $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ as the **coordinates** of $\mathbf{v}$ relative to the basis S. If V has **dimension** n, then every set of n linearly independent vectors in V forms a basis for V. In every application, we have a choice as to what basis we use. In this tutorial, we will desribe the transformation of coordinates of vectors under a change of basis.

We will focus on vectors in R^2, although all of this generalizes to R^n. The **standard basis** in R^2 is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. We specify other bases with reference to this rectangular coordinate system.

Let $B = \{\mathbf{u}, \mathbf{w}\}$ and $B' = \{\mathbf{u}', \mathbf{w}'\}$ be two bases for R^2. For a vector $\mathbf{v} \in V$, given its coordinates $[\mathbf{v}]_B$ in basis B we would like to be able to express $\mathbf{v}$ in tems of its coordinates $[\mathbf{v}]_{B'}$ in basis B', and vice versa.

Suppose the basis vectors $\mathbf{u}'$ and $\mathbf{w}'$ for B' have the following coordinates relative to the basis B:

$$\begin{aligned} [\mathbf{u}']_B &= \begin{bmatrix} a \\ b \end{bmatrix} \\ [\mathbf{w}']_B &= \begin{bmatrix} c \\ d \end{bmatrix}. \end{aligned}$$

This means that

$$\begin{aligned} \mathbf{u}' &= a\mathbf{u} + b\mathbf{w} \\ \mathbf{w}' &= c\mathbf{u} + d\mathbf{w} \end{aligned}$$

The **change of coordinates matrix** from B' to B

$$P = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

governs the change of coordinates of $\mathbf{v} \in V$ under the change of basis from B' to B.

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} [\mathbf{v}]_{B'}.$$

That is, if we know the coordinates of $\mathbf{v}$ relative to the basis B', multiplying this vector by the change of coordinates matrix gives us the coordinates of $\mathbf{v}$ relative to the basis B.

Why?

The transition matrix P is **invertible**. In fact, if P is the change of coordinates matrix from B' to B, the P^{-1} is the change of coordinates matrix from B to B':

$$[\mathbf{v}]_{B'} = P^{-1}[\mathbf{v}]_B$$

Example

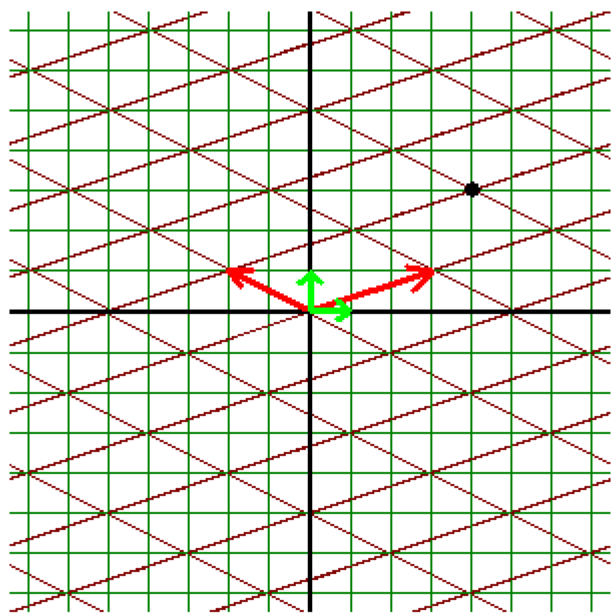
Let $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$. The change of basis matrix form B' to B is

$$P = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}.$$

The vector $\mathbf{v}$ with coordinates $[\mathbf{v}]_{B'} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ relative to the basis B' has coordinates

$$[\mathbf{v}]_B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

relative to the basis B. Since



$$P^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix},$$

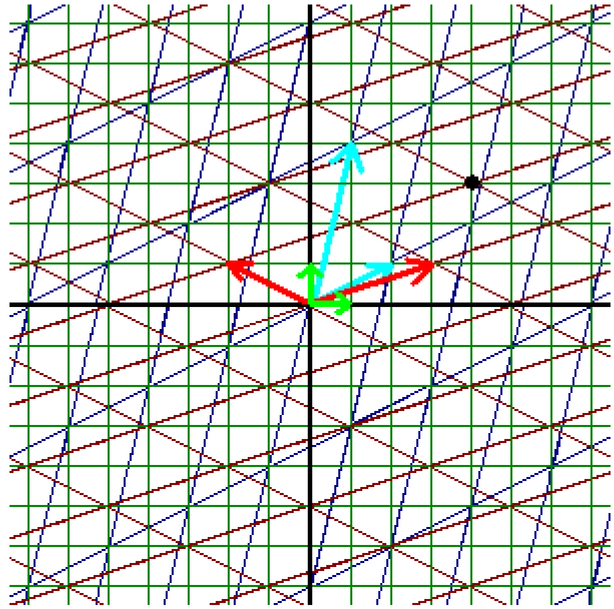
we can verify that

$$[\mathbf{v}]_{B'} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

which is what we started with.

In the following example, we introduce a third basis to look at the relationship between two *non-standard* bases.

Example



Let $B'' = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$. To find the change of coordinates matrix from the basis B' of the previous example to B'', we first express the basis vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ of B' as linear combinations of the basis vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ of B'':

$$\text{Set } \begin{bmatrix} 3 \\ 1 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = c \begin{bmatrix} 2 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

and solve the resulting systems of r $a, b, c,$ and d:

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{11}{7} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{-9}{7} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{4}{7} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Thus, the transition matrix form B' to B'' is

$$\begin{bmatrix} \frac{11}{7} & \frac{-9}{7} \\ \frac{-1}{9} & \frac{4}{7} \end{bmatrix}.$$

The vector $\mathbf{v}$ with coordinates $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ relative to the basis B' has coordinates

$$\begin{bmatrix} \frac{11}{7} & \frac{-9}{7} \\ \frac{-1}{9} & \frac{4}{7} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{13}{7} \\ \frac{2}{7} \end{bmatrix}$$

relative to the basis B''. This is, back in the standard basis,

$$[\mathbf{v}]_B = \frac{13}{7} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2}{7} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix},$$

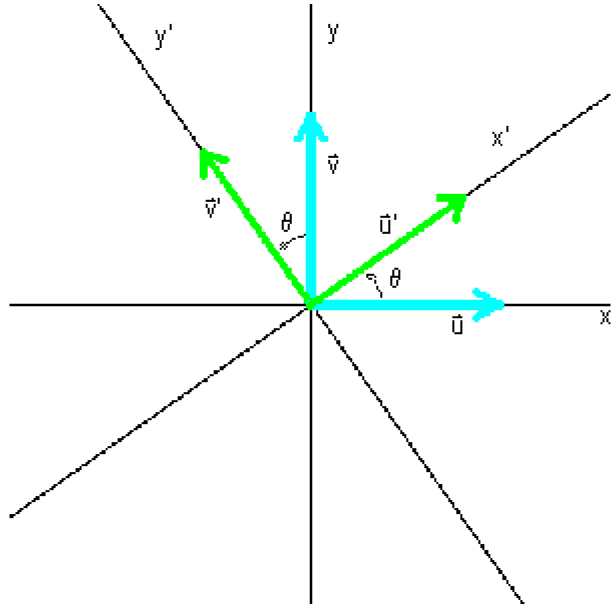
which agrees with the results of the previous example.

Rotation of the Coordinate Axes

Suppose we obtain a new coordinate system from the standard rectangular coordinate system by rotating the axes counterclockwise by an angle θ. The new basis $B' = \{\mathbf{u}', \mathbf{v}'\}$ of unit vectors along the x'- and y'-axes, respectively, has coordinates

$$[\mathbf{u}']_B = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$
$$[\mathbf{v}']_B = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

in the original coordinate system.



Thus, $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. A vector $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]_B$ in the original coordinate system has coordinates $\left[\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right]_{B'}$ given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix}_{B'} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_B$$

in the rotated coordinate system.

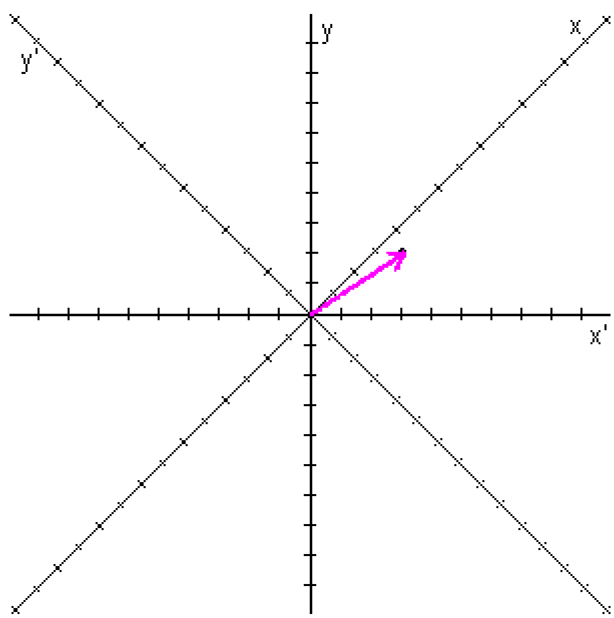
Example

The vector $[\mathbf{v}]_B = \left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right]$ in the original coordinate system has coordinates

$$[\mathbf{v}]_{B'} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

in the coordinate system formed by rotating the axes by 45°.

In the following Exploration, set up your own basis in R^2 and compare the coordinates of vectors in your basis to their coordinates in the standard basis.



Exploration

Key Concepts

Let $B = \{\mathbf{u}, \mathbf{v}\}$ and $B' = \{\mathbf{u}', \mathbf{v}'\}$ be two bases for R^2. If $[\mathbf{u}]_B = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ and $[\mathbf{v}]_B = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$, then $P = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is the **change of coordinates matrix** from B' to B and P^{-1} is the change of coordinates matrix from B to B'. That is, for any $\mathbf{v} \in V$,

$$\begin{aligned} [\mathbf{v}]_B &= P[\mathbf{v}]_{B'} \\ [\mathbf{v}]_{B'} &= P^{-1}[\mathbf{v}]_B. \end{aligned}$$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]