Harvey Mudd College Math Tutorial:

Computing Integrals by Completing the Square

We will review the method of completing the square in the context of evaluating integrals:

Example

Let's start by evaluating

$$\int \frac{dx}{2x^2 - 12x + 26}.$$

The denominator does not factor with rational coefficients, so partial fractions is not a viable option. There is also no obvious substitution to make. Instead, we will complete the square in the denominator to get a recognizable form for the integral.

Now

$$\begin{aligned} 2x^2 - 12x + 26 &= 2[x^2 - 6x + 13] \\ &= 2[(x^2 - 6x + 9) + 4] \\ &= 2[(x-3)^2 + 4]. \end{aligned}$$

Returning to the integral,

$$\begin{aligned} \int \frac{dx}{2x^2 - 12x + 26} &= \int \frac{dx}{2[(x-3)^2 + 4]} \\ &= \frac{1}{2} \int \frac{dx}{(x-3)^2 + 2^2} \\ &= \frac{1}{2} \left[\frac{1}{2} \arctan\left(\frac{x-3}{2} \right) \right] + C \\ &= \frac{1}{4} \arctan\left(\frac{x-3}{2} \right) + C. \end{aligned}$$

Certain other types of integrals can be evaluated by this method as well:

Example

Consider

$$\int \frac{dx}{\sqrt{21 - 4x - x^2}}.$$

Now

$$\begin{aligned} 21 - 4x - x^2 &= 21 - [x^2 + 4x] \\ &= 21 + 4 - [x^2 + 4x + 4] \\ &= 25 - (x+2)^2. \end{aligned}$$

Returning to the integral,

$$\begin{aligned}\int \frac{dx}{\sqrt{21-4x-x^2}} &= \int \frac{dx}{\sqrt{25-(x+2)^2}} \\ &= \arcsin\left(\frac{x+2}{5}\right)+C.\end{aligned}$$

Completing the square is a powerful method that is used to derive the quadratic formula:

We will find the roots of $ax^2+bx+c=0$:

$$\begin{aligned} ax^2+bx+c &= 0 \\ x^2+\frac{b}{a}x+\frac{c}{a} &= 0 \\ x^2+\frac{b}{a}x &= -\frac{c}{a} \\ x^2+\frac{b}{a}x+\frac{b^2}{4a^2} &= \frac{b^2}{4a^2}-\frac{c}{a} \\ \left(x+\frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2} \\ x+\frac{b}{2a} &= \pm\frac{\sqrt{b^2-4ac}}{2a} \\ x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \end{aligned}$$

which is the familiar quadratic formula!

Key Concept

By completing the square, we may rewrite any quadratic polynomial

$$ax^2+bx+x$$

in the form

$$a\left[(x+k_1)^2+k_2\right]$$

where k_1 and k_2 may be positive or negative. Integrals containing negative or non-integer powers of ax^2+bx+c can often be computed using a trigonometric substitution or looked up in an integral table after being rewritten in this form.

[I'm ready to take the quiz.] [I need to review more.]
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