Harvey Mudd College Math Tutorial:

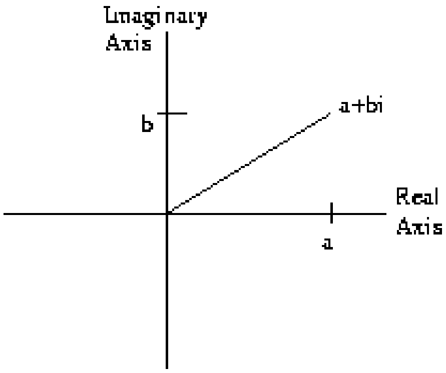
# Complex Numbers

The complex numbers are an extension of the real numbers containing all roots of quadratic equations. If we define $i$ to be a solution of the equation $x^2 = -1$, then the set $\mathbb{C}$ of complex numbers is represented in **standard form** as

$$\{a + bi | a, b \in R\}.$$

We often use the variable $z = a + bi$ to represent a complex number. The number $a$ is called the **real part** of $z$ (Re $z$) while $b$ is called the **imaginary part** of $z$ (Im $z$). Two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

We represent complex numbers graphically by associating $z = a + bi$ with the point $(a, b)$ on the **complex plane**.



## Basic Operations

The basic operations on complex numbers are defined as follows:

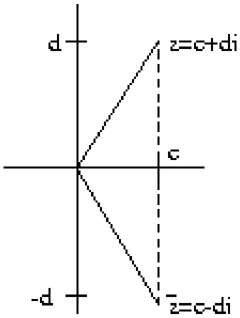
$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i\\(a+bi)-(c+di) &= (a-c)+(b-d)i\\(a+bi)(c+di) &= ac+adi+bci+bdi^2\\ &= (ac-bd)+(bc+ad)i\end{aligned}$$

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i$$

In dividing $a+bi$ by $c+di$, we rationalized the denominator using the fact that $(c+di)(c-di) = c^2 - cdi + cdi - d^2i^2 = c^2 + d^2$. The complex numbers $c+di$ and $c-di$ are called **complex conjugates**. If $z = c+di$, we use $\overline{z}$ to denote $c - di$.
Viewing $z = a + bi$ as a vector in the complex plane, it has magnitude

$$|z| = \sqrt{a^2 + b^2},$$

which we call the **modulus** or **absolute value** of $z$. Notice that $z\overline{z} = |z|^2$.



**Examples**

- $(2+3i)(2-3i) = 4 - 6i + 6i - 9i^2 = 4 + 9 = 13.$
- $|2+3i| = |2-3i| = \sqrt{4+9} = \sqrt{13}.$

## Polar Form

For $z = a + bi$, let

$$\begin{aligned} a &= r\cos\theta \\ b &= r\sin\theta \end{aligned}$$

from which we can also obtain

$$\begin{aligned} r &= \sqrt{a^2+b^2} = |z| \\ \tan\theta &= \frac{b}{a}. \end{aligned}$$

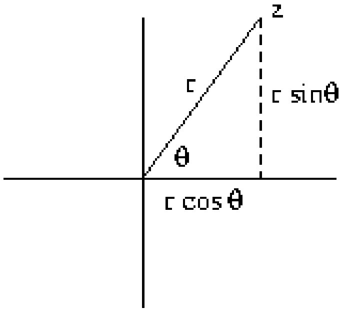
Then

$$z = r\cos\theta + ir\sin\theta$$

and so, by Euler's Equation, we obtain the **polar form**

$$z = re^{i\theta}.$$

Here, $r$ is the magnitude of $z$ and $\theta$ is called the **argument** of $z$ (arg $z$). The argument is not unique; we can add multiples of $2\pi$ to $\theta$ without changing $z$. We define Arg $z$, the **principal value** of the argument, to be in $(-\pi, \pi]$. The principal value is unique for each $z$ but creates unavoidable (yet interesting!) complications due to its discontinuity across the negative real axis where it jumps from $\pi$ to $-\pi$. This jump is called a **branch cut**.



If you write

$$\theta = \tan^{-1}\frac{y}{x},$$

be careful to choose the value for $\theta$ in the correct quadrant.

**Euler's Equation:**

$$e^{i\theta} = \cos\theta + i\sin\theta$$

**Examples**

- $e^{i\pi} = \cos\pi + i\sin\pi = -1$
- $3e^{i\pi/2} = 3(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 3i$
- $2e^{i\pi/6} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{3} + i$

Multiplication and division of complex numbers is amazingly simple in polar form! If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

If $z = re^{i\theta}$, then $\overline{z} = re^{-i\theta}$ (Do you see why?) and so $z\overline{z} = (re^{i\theta})(re^{-i\theta}) = r^2$.

**Example**

To calculate $(1+i)^8$, we can first rewrite $1+i$ as $\sqrt{2}e^{i\pi/4}$. Then

$$\begin{aligned} \left(\sqrt{2}e^{i\pi/4}\right)^8 &= (\sqrt{2})^8 e^{i8\pi/4} \\ &= 16e^{2\pi i} \\ &= 16. \end{aligned}$$

$$\begin{aligned} \sqrt{1^2+1^2} &= \sqrt{2} \\ \tan^{-1}\left(\frac{1}{1}\right) &= \frac{\pi}{4} \end{aligned}$$

**Roots of Unity**

The equation

$$z^n = 1$$

has $n$ complex-valued solutions, called the $n^{th}$ roots of unity. Since we know each root has magnitude 1, let $z = e^{i\theta}$. Then

$$\begin{aligned} (e^{i\theta})^n &= 1 \\ e^{in\theta} &= e^{i(2\pi k)} \\ n\theta &= 2\pi k \\ \theta &= \frac{2\pi k}{n} \end{aligned}$$

so the $n^{th}$ roots of unity are of the form

$$z = e^{i\frac{2\pi k}{n}}.$$

There are $n$ distinct roots, after which we start dumplicating roots already found.

$(e^{i\theta})^n = e^{in\theta}$ together with Euler's Equation, gives us **deMoivre's Formula:**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$1 = e^{0i} = e^{2\pi ki}$$

for $k = 0, \pm 1, \pm 2, \ldots$

These are evenly spaced around the unit circle

**Example**
The $3rd$ roots of unity are

$$\begin{array}{rcl} & 1 & \\ e^{i\frac{2\pi}{3}} & = & -\frac{1}{2}+i\frac{\sqrt{3}}{2} \\ e^{-i\frac{2\pi}{3}} & = & -\frac{1}{2}-i\frac{\sqrt{3}}{2} \end{array}$$

You can verify that $(-\frac{1}{2}+i\frac{\sqrt{3}}{2})^3 = 1$ and $(-\frac{1}{2}-i\frac{\sqrt{3}}{2})^3 = 1$.

This tutorial has reviewed the basics of complex arithmetic. The methods of complex analysis, which build on this background, are both intriguing and powerful!

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## Key Concept

**Standard Form**

$$\begin{array}{rcl} z & = & a+bi \\ a & = & \text{Re } z \\ b & = & \text{Im } z \\ |z| & = & \sqrt{a^2+b^2} \\ \overline{z} & = & a-bi \end{array}$$

$$\begin{array}{rcl} a & = & r\cos\theta \\ b & = & r\sin\theta \\ r & = & \sqrt{a^2+b^2} \\ \tan\theta & = & \frac{b}{a} \end{array}$$

**Polar Form**

$$\begin{array}{rcl} z & = & re^{i\theta} \\ r & = & |z| \\ \theta & = & \arg\ z \\ \overline{z} & = & re^{-i\theta} \end{array}$$

Euler's Equation,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

provides the connection between these two representations of complex numbers.

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]