Harvey Mudd College Math Tutorial:

Differentiating Special Functions

In this tutorial, we review the differentiation of trigonometric, logarithmic, and exponential functions.

Trigonometric Functions

The derivatives of the basic trigonometric functions are given here for reference.

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\cot x$	$-\csc^2 x$

The derivatives of $\sin x$ and $\cos x$ can be derived using the **limit definition of the derivative**. For $\sin x$,

$$\begin{aligned} \frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\ &= \lim_{h\to 0} \left[\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}\right] \\ &= \sin x \lim_{h\to 0}\left[\frac{\cos h - 1}{h}\right] + \cos x \lim_{h\to 0}\left[\frac{\sin h}{h}\right] \\ &= \sin x(0) + \cos x(1) \\ &= \cos x. \end{aligned}$$

The derivative of $\cos x$ is derived analogously. Then the remaining derivatives can be derived using the quotient rule, since all the other trigonometric functions are quotients involving $\sin x$ and $\cos x$.

Example

The derivative of $\tan(x^2)$ is $\sec^2(x^2) \cdot \frac{d}{dx}(x^2) = 2x\sec^2(x^2)$ by the **chain rule**.

Logarithmic Functions

By the definition of the natural logarithm, $\frac{d}{dx}[\ln x] = \frac{1}{x}$ for $x > 0$. Also, $\frac{d}{dx}[\ln|x|] = \frac{1}{x}$ for all $x \neq 0$. To see this, suppose $x < 0$. Then $\ln|x| = \ln(-x)$.

So

$$\begin{aligned} \frac{d}{dx}[\ln|x|] &= \frac{d}{dx}[\ln(-x)] \\ &= \frac{d}{dx}(-x)\left(\frac{1}{-x}\right) \\ &= (-1)\left(\frac{1}{-x}\right) \\ &= \frac{1}{x}. \end{aligned}$$

Example

By the **chain rule**, the derivative of $\ln(x^3+5)$ is $\frac{d(x^3+5)}{dx} \cdot \frac{1}{x^3+5} = \frac{3x^2}{x^3+5}$.

Exponential Functions

There is an elegant way to show that $\frac{d}{dx}[e^x] = e^x$. We start with the identity $\ln(e^x) = x$. Differentiating both sides,

$$\begin{aligned} \frac{d}{dx}[\ln(e^x)] &= \frac{d}{dx}(x) \\ \frac{d}{dx}[\ln(e^x)] &= 1 \\ \frac{d}{dx}(e^x) \cdot \frac{1}{e^x} &= 1 \\ \frac{d}{dx}(e^x) &= e^x. \end{aligned}$$

Since e^x is never 0, this derivation holds for all x.

Example

The derivative of e^{-3x+2} is $e^{-3x+2} \cdot \frac{d}{dx}(-3x+2) = -3e^{-3x+2}$.

Key Concepts

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\cot x$	$-\csc^2 x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

[I'm ready to take the quiz.] [I need to review more.]
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