

The First Derivative: Maxima and Minima

Consider the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 3$$

on the interval $[-2, 3]$. We cannot find regions of which f is increasing or decreasing, relative maxima or minima, or the absolute maximum or minimum value of f on $[-2, 3]$ by inspection. Graphing by hand is tedious and imprecise. Even the use of a graphing program will only give us an approximation for the locations and values of maxima and minima. We can use the first derivative of f , however, to find all these things quickly and easily.

Increasing or Decreasing?

Let f be continuous on an interval I and differentiable on the interior of I .

- If $f'(x) > 0$ for all $x \in I$, then f is *increasing* on I .
- If $f'(x) < 0$ for all $x \in I$, then f is *decreasing* on I .

Example

The function $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ has first derivative

$$\begin{aligned} f'(x) &= 12x^3 - 12x^2 - 24x \\ &= 12x(x^2 - x - 2) \\ &= 12x(x + 1)(x - 2). \end{aligned}$$

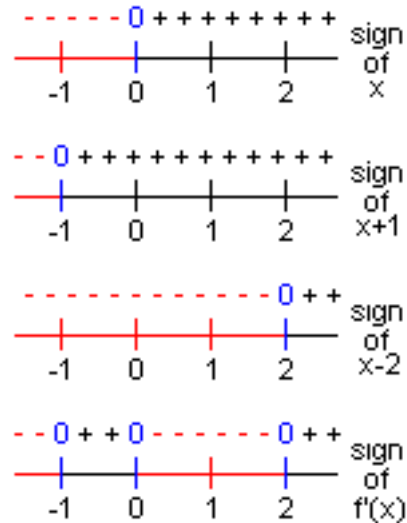
Thus, $f(x)$ is increasing on $(-1, 0) \cup (2, \infty)$ and decreasing on $(-\infty, -1) \cup (0, 2)$.

Relative Maxima and Minima

Relative extrema of f occur at **critical points** of f , values x_0 for which either $f'(x_0) = 0$ or $f'(x_0)$ is undefined.

First Derivative Test

Suppose f is continuous at a critical point x_0 .



- If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0 , then f has a relative maximum at x_0 .
- If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0 , then f has a relative minimum at x_0 .
- If $f'(x)$ has the same sign on both an open interval extending left from x_0 and an open interval extending right from x_0 , then f does not have a relative extremum at x_0 .

In summary, relative extrema occur where $f'(x)$ changes sign.

Example

Our function $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ is differentiable everywhere on $[-2, 3]$, with $f'(x) = 0$ for $x = -1, 0, 2$. These are the three critical points of f on $[-2, 3]$. By the First Derivative Test, f has a relative maximum at $x = 0$ and relative minima at $x = -1$ and $x = 2$.



Absolute Maxima and Minima

- If f has an extreme value on an *open* interval, then the extreme value occurs at a critical point of f .
- If f has an extreme value on a *closed* interval, then the extreme value occurs either at a critical point or at an endpoint.

According to the **Extreme Value Theorem**, if a function is continuous on a closed interval, then it achieves both an absolute maximum and an absolute minimum on the interval.

Example

Since $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ is continuous on $[-2, 3]$, f must have an absolute maximum and an absolute minimum on $[-2, 3]$. We simply need to check the value of f at the critical points $x = -1, 0, 2$ and at the endpoints $x = -2$ and $x = 3$:

$$\begin{aligned}
 f(-2) &= 35, \\
 f(-1) &= -2, \\
 f(0) &= 3, \\
 f(2) &= -29, \\
 f(3) &= 30.
 \end{aligned}$$

Thus, on $[-2, 3]$, $f(x)$ achieves a maximum value of 35 at $x = -2$ and a minimum value of -29 at $x = 2$.

We have discovered a lot about the shape of $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ without ever graphing it! Now take a look at the graph and verify each of our conclusions.

Graph it!

Key Concepts

- **Increasing or Decreasing?**

Let f be continuous on an interval I and differentiable on the interior of I . If $f'(x) > 0$ for all $x \in I$, then f is *increasing* on I . If $f'(x) < 0$ for all $x \in I$, then f is *decreasing* on I .

- **Relative Maxima and Minima**

By the First Derivative Test, relative extrema occur where $f'(x)$ changes sign.

- **Absolute Maxima and Minima**

If f has an extreme value on a closed interval, then the extreme value occurs either at a critical point or at an endpoint.

[I'm ready to take the quiz.] [I need to review more.]
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