Harvey Mudd College Math Tutorial:

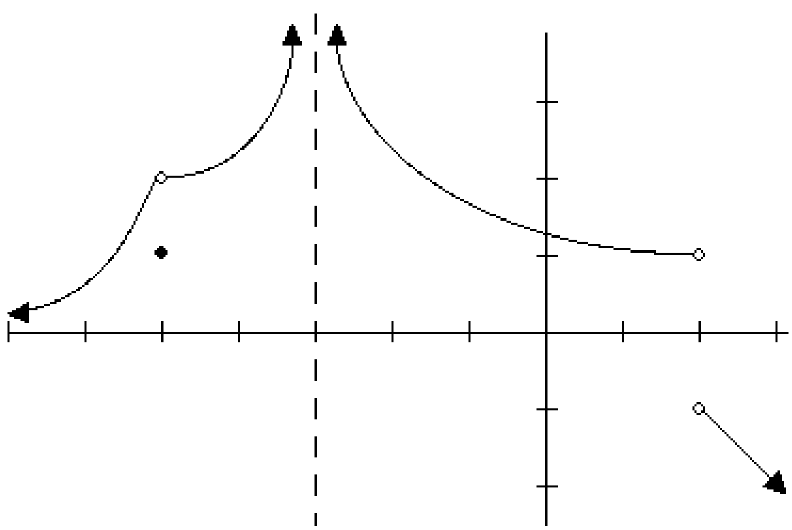
Computing Limits

Intuitively, we say that $\lim_{x \to c} f(x) = L$ if f is defined near (but not necessarily at) c and $f(x)$ approaches L as x approaches c.

If we let x approach c from the left side only, we write $\lim_{x \to c^-} f(x)$ since x is approaching c from smaller values. Similarly, for x approaching c from the right, we write $\lim_{x \to c^+} f(x)$. The *two-sided* limit $\lim_{x \to c} f(x)$ exists if and only if both of these *one-sided* limits exist and are equal.

An Intuitive Example

Consider the graph of a function $f(x)$ shown below.



Evaluate each of the following. Answers can be found on the last page.

1. $f(2)$
2. $\lim_{x \to 2^+} f(x)$
3. $\lim_{x \to 2^-} f(x)$
4. $\lim_{x \to 2} f(x)$
5. $f(-5)$
6. $\lim_{x \to -5^+} f(x)$
7. $\lim_{x \to -5^-} f(x)$
8. $\lim_{x \to -5} f(x)$
9. $\lim_{x \to -3} f(x)$
10. $\lim_{x \to -\infty} f(x)$
11. $\lim_{x \to \infty} f(x)$
12. $\lim_{x \to c} f(x)$ for $c \neq -5, -3, 2$.

Definition of the Limit

More rigorously, let f be defined at all x in an open interval containing c, except possibly at c itself.

Then

$$\lim_{x \to c} f(x) = L$$

if and only if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if } 0 < |x - c| < \delta \text{ then } |f(x) - L| < \varepsilon.$$

In words, $\lim_{x \to c} f(x) = L$ if and only if by taking x close enough to c we can get $f(x)$ arbitrarily close to L.

Figures   Exploration

Properties of the Limit

Each of the following properties is proven using the rigorous definition of the limit. Let lim stand for $\lim_{x \to c}$, $\lim_{x \to c^+}$, or $\lim_{x \to c^-}$. Assume $\lim f(x)$ and $\lim g(x)$ both exist.

- **(Uniqueness)** If $\lim f(x) = L_1$ and $\lim f(x) = L_2$, then $L_1 = L_2$.
- **(Addition)** $\lim[f(x) + g(x)] = \lim f(x) + \lim g(x)$.
- **(Scalar multiplication)** $\lim[\alpha f(x)] = \alpha \lim f(x)$.
- **(Multiplication)** $\lim[f(x)g(x)] = \lim f(x) \cdot \lim g(x)$.
- **(Division)** $\lim \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim g(x)}$, provided $\lim g(x) \neq 0$.
- **(Powers)** $\lim[f(x)]^n = [\lim f(x)]^n$ for any positive integer n.

In practice, much of the time we can "reason out" the value of a limit without explicitly using the ε–δ definition.

Examples

- $\lim_{x \to 2} \sqrt{x^2 + 12} = 4$ since the function $f(x) = \sqrt{x^2 + 12}$ is continuous at $x = 2$ and $f(2) = 4$.
- $\lim_{x \to \infty} \dfrac{1}{x} = 0$ since as x increases, $\dfrac{1}{x}$ gets arbitrarily close to 0.

- $\lim_{x\to 0^+} \ln|x|$ tends to $-\infty$ and so does not exist since as x decreases to 0, $\ln|x|$ gets arbitrarily large in magnitude and negative.
- $\lim_{x\to 3} \frac{x^2-9}{x-3} = 6$ even though $f(x) = \frac{x^2-9}{x-3}$ is undefined at 3 since $\frac{x^2-9}{x-3} = x+3$ and $\lim_{x\to 3} x+3 = 6$.

What about something like $\lim_{x\to 0} \frac{\sin x}{x}$? When we cannot easily "reason out" the value of a limit, we can often use numerical methods or L'Hôpital's Rule to determine the value of the limit. Can you convince yourself that $\lim_{x\to 0} \frac{\sin x}{x} = 1$?

Key Concepts

Let f be defined at all x in an open interval containing c, except possibly at c itself.

Then

$$\lim_{x\to c} f(x) = L$$

if and only if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if } 0 < |x-c| < \delta \text{ then } |f(x) - L| < \varepsilon.$$

In words, $\lim_{x\to c} f(x) = L$ if and only if by taking x close enough to c we can get $f(x)$ arbitrarily close to L.

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]

Answers to questions on page 1.

1. undefined
2. -1
3. 1
4. undefined
5. 1
6. 2
7. 2
8. 2
9. undefined (∞)
10. 0
11. undefined ($-\infty$)
12. $f(c)$