Harvey Mudd College Math Tutorial:

Geometry of Linear Transformations of the Plane

Let V and W be **vector spaces.** Recall that a function $T : V \to W$ is called a **linear transformation** if it preserves both vector addition and scalar multiplication:

$$\begin{aligned} T(\mathbf{v_1} + \mathbf{v_2}) &= T(\mathbf{v_1}) + T(\mathbf{v_2}) \\ T(r\mathbf{v_1}) &= rT(\mathbf{v_1}) \end{aligned}$$

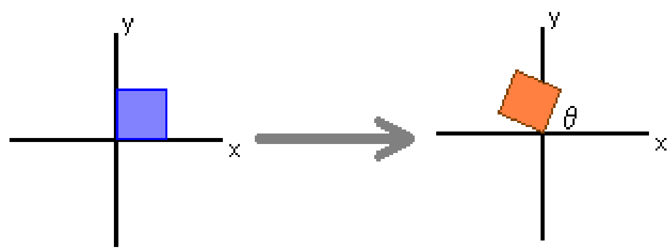
for all $\mathbf{v_1}, \mathbf{v_2} \in V$.

If $V = R^2$ and $W = R^2$, then $T : R^2 \to R^2$ is a linear transformation if and only if there exists a 2×2 matrix A such that $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in R^2$. Matrix A is called the **standard matrix** for T. The columns of A are $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$, respectively. Since each linear transformation of the plane has a unique standard matrix, we will identify linear transformations of the plane by their standard matrices. It can be shown that if A is invertible, then the linear transformation defined by A maps parollelograms to parallelograms. We will often illustrate the action of a linear transformation $T : R^2 \to R^2$ by looking at the image of a unit square under T.

Rotations

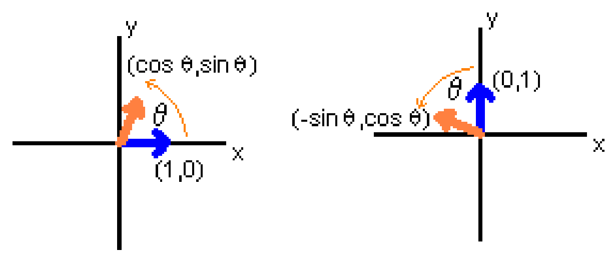
The standard matrix for the linear transformation $T : R^2 \to R^2$ that rotates vectors by an angle θ is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$



This is easily drived by noting that

$$\begin{aligned} T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) &= \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \\ T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) &= \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}. \end{aligned}$$

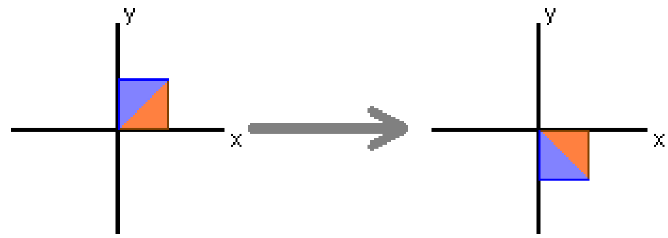


Reflections

For every line in the plane, there is a linear transformation that reflects vectors about that line. Relection about the x-axis is given by the standard matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

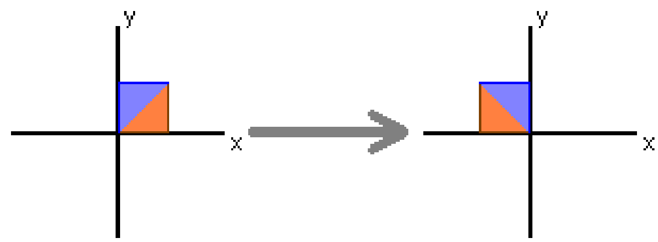
which takes the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x \\ -y \end{bmatrix}$.



Reflection about the y-axis is given by the standard matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

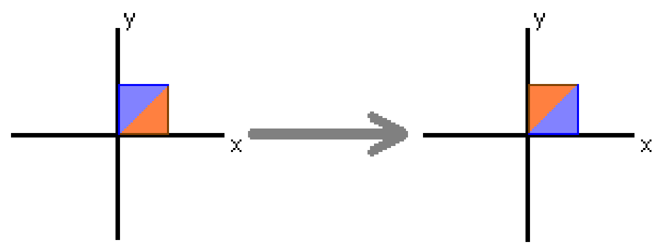
taking $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} -x \\ y \end{bmatrix}$.



Finally, reflection about the line $y = x$ is given by

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and takes the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} y \\ x \end{bmatrix}$.

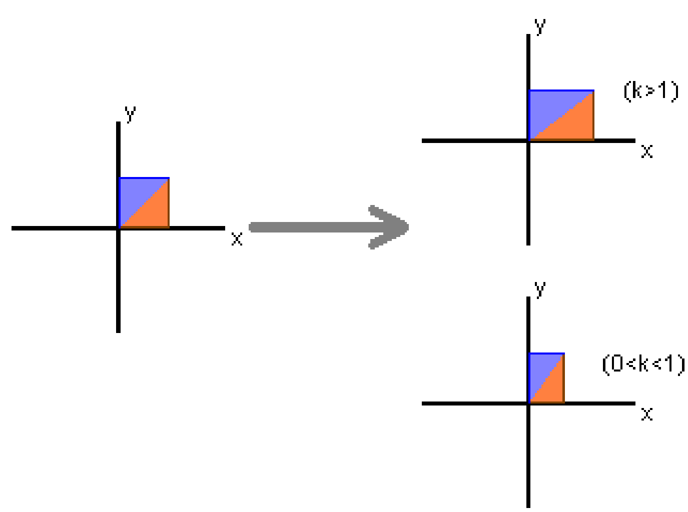


Expansions and Compressions

The standard matrix

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

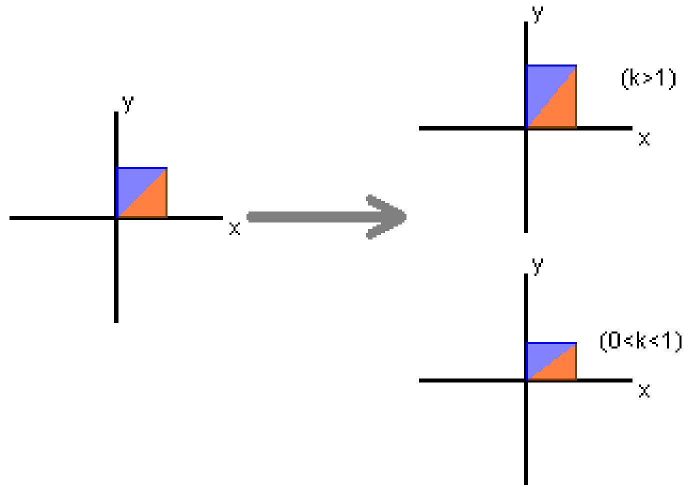
"stretches" the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ along the x-axis to $\begin{bmatrix} kx \\ y \end{bmatrix}$ for $k > 1$ and "compresses" it along the x-axis for $0 < k < 1$.



Similarlarly,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

stretches or compresses vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x \\ ky \end{bmatrix}$ along the y-axis.

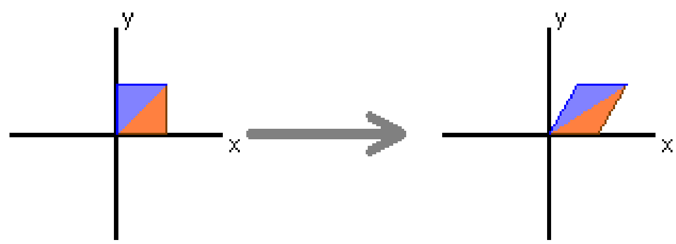


Shears

The standard matrix

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

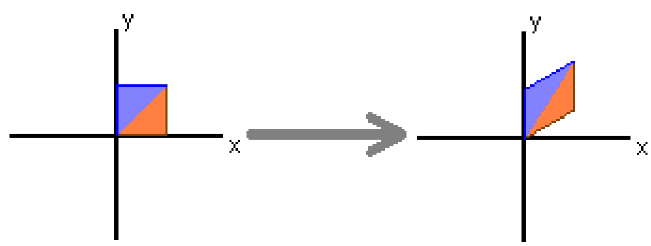
taking vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x+ky \\ y \end{bmatrix}$ is called a **shear in the x-direction**.



Similarly,

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

takes vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x \\ y+kx \end{bmatrix}$ and is called a **shear in the y-direction**.



Notes

- If finitely many linear transformations from R^2 to R^2 are performed in succession, then there exists a single linear transformation with thte same effect.
- If the standard matrix for a linear transformation $T : R^2 \to R^2$ is **invertible**, then it can be shown that the geometric effect of T is the same as some sequence of reflections, expansions, compressions, and shears.

In the following Exploration, you can investigate the connection between the entries in a standard matrix and the effect the corresponding linear transformation has geometrically.

Key Concept

For every linear transformation $T : R^2 \to R^2$ of the plane, there exists a standard matrix A such that

$$T(\mathbf{v}) = A\mathbf{v} \text{ for all } \mathbf{v} \in R^2.$$

Every linear transformation of the plane with an *invertible* standard matrix has the geometric effect of a sequence of reflections, expansions, compressions, and shears.

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]