Harvey Mudd College Math Tutorial:

Lines, Planes, and Vectors

In this tutorial, we will use vector methods to represent lines and planes in 3-space.

Displacement Vector

The displacement vector $\mathbf{v}$ with initial point (x_1, y_1, z_1) and terminal point (x_2, y_2, z_2) is

$$\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

Why?

That is, if vector $\mathbf{v}$ were positioned with its initial point at the origin, then its terminal point would be at $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Example

The vector $\mathbf{v}$ with initial point $(-1, 4, 5)$ and final point $(4, -3, 2)$ is

$$\mathbf{v} = (4 - (-1), -3 - 4, 2 - 5) = (5, -7, -3).$$

Parametric Equations for a Line in 3-space

The line throught the point (x_0, y_0, z_0) and parallel to the non-zero vector $\mathbf{v} = (a, b, c)$ has parametric equations

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct. \end{aligned}$$

Why?

Example

The line through $(2, -1, 3)$ and parallel to the vector $\mathbf{v} = (3, -7, 4)$ has parametric equations

$$\begin{aligned} x &= 2 + 3t \\ y &= -1 - 7t \\ z &= 3 + 4t. \end{aligned}$$

Notice that when $t = 0$, we are at the point $(2, -1, 3)$. As t increases or decreases from 0, we move away from this point parallel to the direction indicated by $(3, -7, 4)$.

If you know two points $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ that a line passes through, you can find a parametrization for the lilne. First, find the displacement vector $\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. then write down parametric equations for the line through either p_1 or p_2 and parallel to $\mathbf{v}$.

Equation of a Plane in 3-space

The equation of the plane containing the point (x_0, y_0, z_0) with normal vector $\mathbf{n} = (a, b, c)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Why?

Thus, the graph of the equation

$$ax + by + cz = d$$

is a plane with normal vector (a, b, c).

Example

The equation of the plane containing $(2, 4, -1)$ and normal to the vector $\mathbf{n} = (3, 5, -2)$ is

$$3(x - 2) + 5(y - 4) - 2(z - (-1)) = 0.$$

Simplifying,

$$3x + 5y - 2z = 28.$$

With a little extra work, we can use this procedure to find the equation of the plane defined by any thee points. First, compute displacement vectors $\mathbf{u}$ and $\mathbf{v}$ between two pairs of these points. Then $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ in normal to the plane. Now, use one of the points and the vector $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ to obtain the equation of the plane.

Key Concepts

- **Displacement Vector**

  The displacement vector $\mathbf{v}$ with initial point (x_1, y_1, z_1) and terminal point (x_2, y_2, z_2) is $\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

- **Parametric Equations for a line in 3-space**

  The line throught he point (x_0, y_0, z_0) and parallel to the non-zero vector $\mathbf{v} = (a, b, c)$ has parametric equations

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct. \end{aligned}$$

- **Equation of a plane in 3-space**

  The equation of the plane containing the point (x_0, y_0, z_0) with normal vector $\mathbf{n} = (a, b, c)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]