Harvey Mudd College Math Tutorial:

Parametric Equations

Think of a curve being traced out over time, sometimes doubling back on itself or crossing itself. Such a curve cannot be described by a function $y = f(x)$. Instead, we will describe our position along the curve at time t by

$$\begin{aligned} x &= x(t) \\ y &= y(t). \end{aligned}$$

Then x and y are related to each other through their dependence on the **parameter** t.

Example

Suppose we trace out a curve according to

$$\begin{aligned} x &= t^2 - 4t \\ y &= 3t \end{aligned}$$

where $t \geq 0$. The arrow on the curve indicates the direction of increasing time or **orientation** of the curve. Click the link above for a java applet of this example. Drag the box along the curve and notice how x and y vary with t.

The parameter does not always represent time:

Example

Consider the parametric equation

$$\begin{aligned} x &= 3\cos\theta \\ y &= 3\sin\theta. \end{aligned}$$

Here, the parameter θ represents the polar angle of the position on a circle of radius 3 centered at the origin and oriented counterclockwise.

Differentiating Parametric Equations

Let $x = x(t)$ and $y = y(t)$. Suppose for the moment that we are able to re-write this as $y(t) = f(x(t))$. Then $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$ by the Chain Rule. Solving for $\dfrac{dy}{dx}$ and assuming $\dfrac{dx}{dt} \neq 0$,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

a formula that holds in general.

Example

If $x = t^2 - 3$ and $y = t^8$, then $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 8t^7$. So

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{8t^7}{2t} = 4t^6 \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d[\frac{dy}{dx}]}{dt}}{\frac{dx}{dt}} = \frac{24t^5}{2t} = 12t^4. \end{aligned}$$

Notes

- It is often possible to re-write the parametric equations without the parameter. In the second example, $\frac{x}{3} = \cos\theta$, $\frac{y}{3} = \sin\theta$. Since $\cos^2\theta + \sin^2\theta = 1$, $\left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$. Then $x^2 + y^2 = 9$, which is the equation of a circle as expected. When you do eliminate the parameter, always check that you have not introduced extraneous portions of the curve.

- Every curve has infinitely many parametrizations, amounting to different scales for the parameter. For example,

$$\begin{aligned} x &= 3\cos 2\theta \\ y &= 3\sin 2\theta \end{aligned}$$

  traces out the circle from the second example twice as “quickly,” completing a full revolution in π rather than 2π units of θ.

- Every equation $y = f(x)$ may be re-written in parametric form by letting $x = t$, $y = f(t)$.

Key Concepts

A curve in the xy-plane may be described by a pair of parametric equations

$$x = x(t)$$

$$y = y(t)$$

where x and y are related through their dependence on t. This is particularly useful when neither x nor y is a function of the other.

The derivative of y with respect to x (in terms of the parameter t) is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

[I'm ready to take the quiz.] [I need to review more.]
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