Harvey Mudd College Math Tutorial:

Review of Trigonometric, Logarithmic, and Exponential Functions

In this tutorial, we review trigonometric, logarithmic, and exponential functions with a focus on those properties which will be useful in future math and science applications.

Trigonometric Functions

Geometrically, there are two ways to describe trigonometric functions:

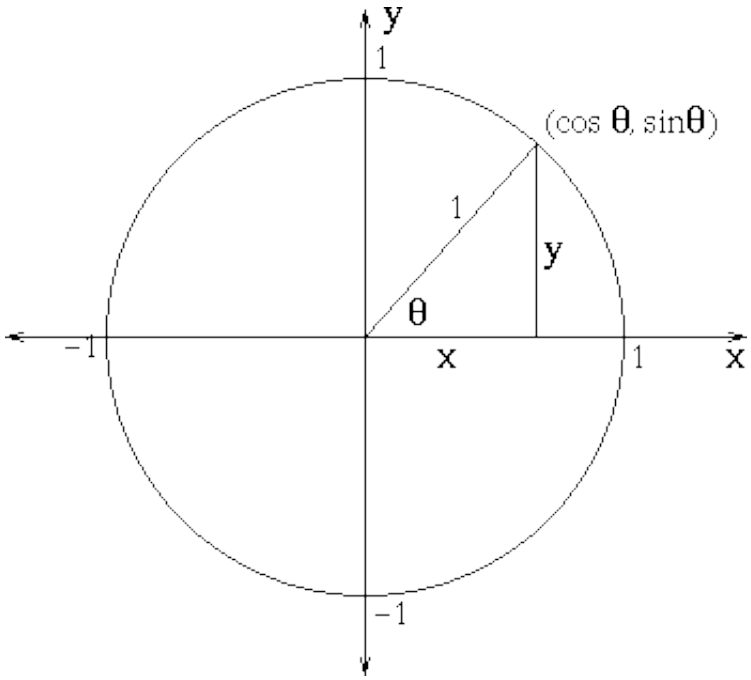
Polar Angle

$x = \cos\theta$
$y = \sin\theta$

Measure θ in radians:
$\theta = \dfrac{\text{arc length}}{\text{radius}}$

For example, $\quad 180^\circ = \dfrac{\pi r}{r} = \pi$ radians

$\text{Radians} = \dfrac{\text{degrees}}{180} \cdot \pi$



Right Triangle

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$$

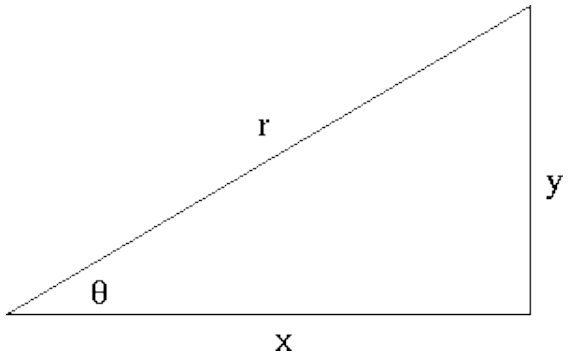
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$$

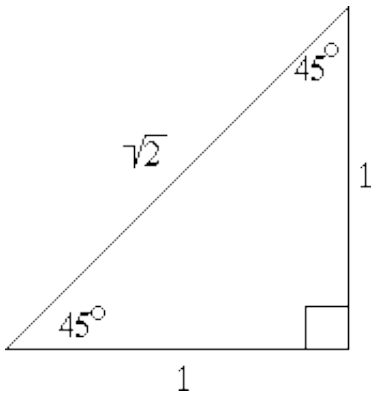
$$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$$

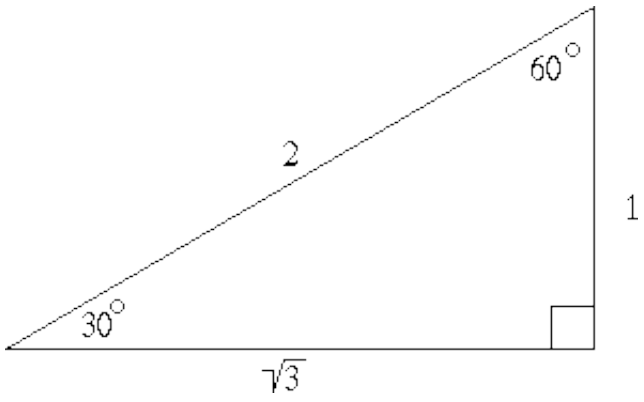
$$\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$$



Evaluating Trigonometric Functions

	0 rad 0°	$\pi/6$ rad 30°	$\pi/4$ rad 45°	$\pi/3$ rad 60°	$\pi/2$ rad 90°
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undefined





$$\sin(-\theta) = -\sin\theta \qquad \sin(\theta+\pi/2) = \cos\theta$$
$$\cos(-\theta) = \cos\theta \qquad \cos(\theta+\pi/2) = -\sin\theta$$

$$\cos(\theta+\pi) = -\cos\theta \qquad \cos(\theta+2\pi) = \cos\theta$$
$$\sin(\theta+\pi) = -\sin\theta \qquad \sin(\theta+2\pi) = \sin\theta$$

Trigonometric Identities

We list here some of the most commonly used identities:

1. $\cos^2\theta + \sin^2\theta = 1$

2. $\cos^2\theta = \dfrac{1}{2}[1 + \cos(2\theta)]$

3. $\sin^2\theta = \dfrac{1}{2}[1 - \cos(2\theta)]$

4. $\sin(2\theta) = 2\sin\theta\cos\theta$

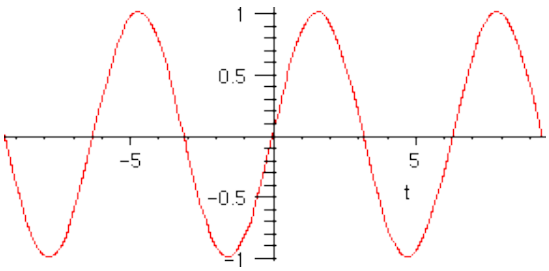
5. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

6. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

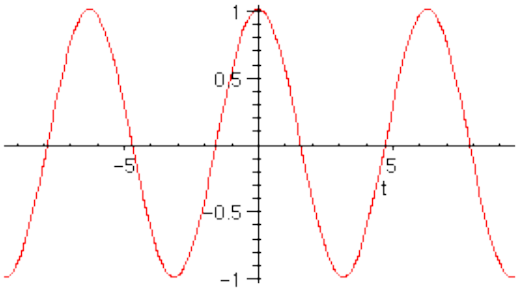
7. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

8. $C_1\cos(\omega x) + C_2\sin(\omega x) = A\sin(\omega x + \phi)$
where $A = \sqrt{C_1^2 + C_2^2}$, $\quad \phi = \arctan(C_1/C_2)$

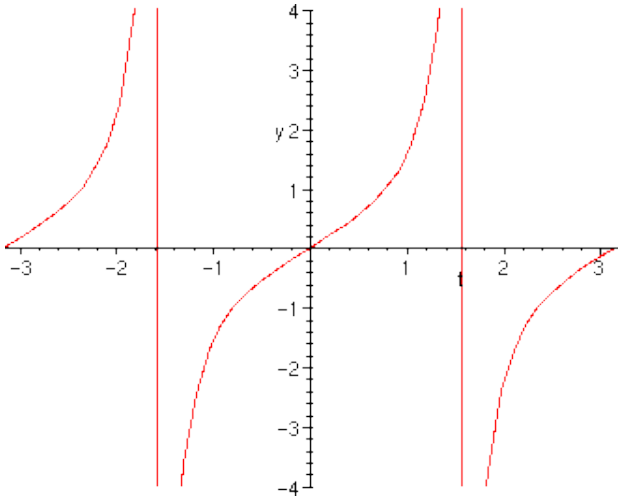
Graphs of Trigonometric Functions



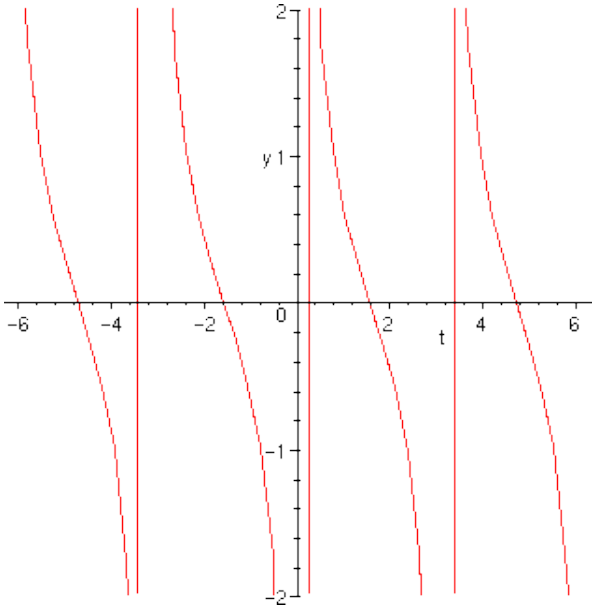
$\sin x$



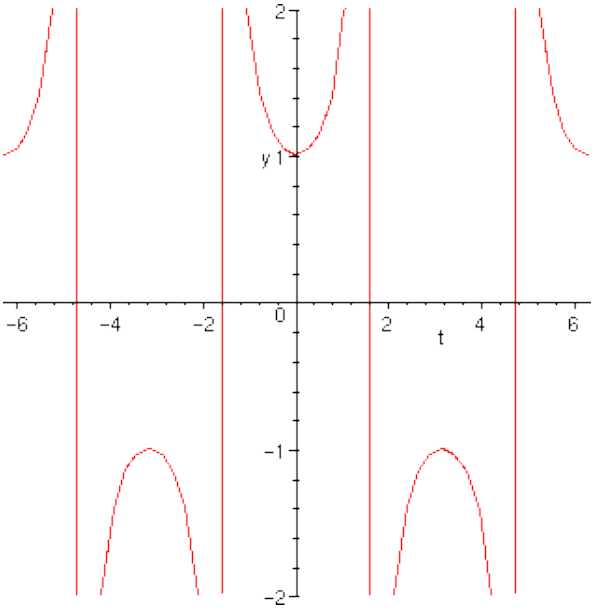
$\cos x$



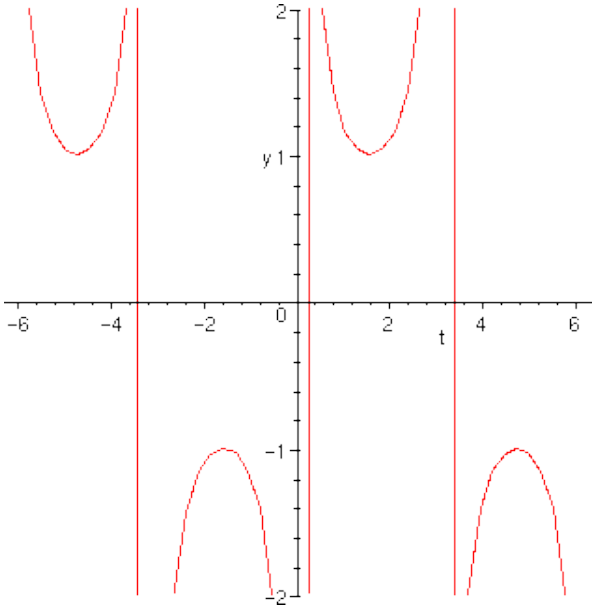
$\tan x$



$\cot x$



sec x



csc x

Logarithmic and Exponential Functions

Logarithmic and exponential functions are inverses of each other:

$$\begin{aligned} y = \log_b x \quad &\text{if and only if} \quad x = b^y \\ y = \ln x \quad &\text{if and only if} \quad x = e^y. \end{aligned}$$

In words, $\log_b x$ is the exponent you put on base b to get x. Thus,

$$log_b b^x = x \qquad \text{and} \qquad b^{\log_b x} = x.$$

More Properties of Logarithmic and Exponential Functions

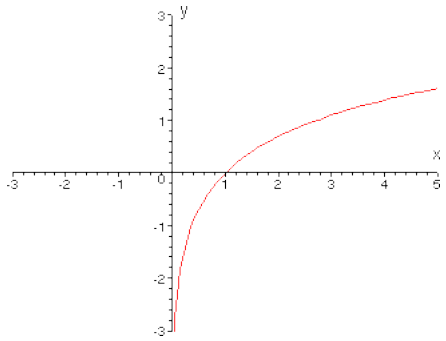
Notice the relationship between each pair of identities:

$$\log_b 1 = 0 \quad \longleftrightarrow \quad b^0 = 1 \qquad \log_b ac = \log_b a + \log_b c \quad \longleftrightarrow \quad b^m b^n = b^{m+n}$$

$$\log_b b = 1 \quad \longleftrightarrow \quad b^1 = b \qquad \log_b \frac{a}{c} = \log_b a - \log_b c \quad \longleftrightarrow \quad \frac{b^m}{b^n} = b^{m-n}$$

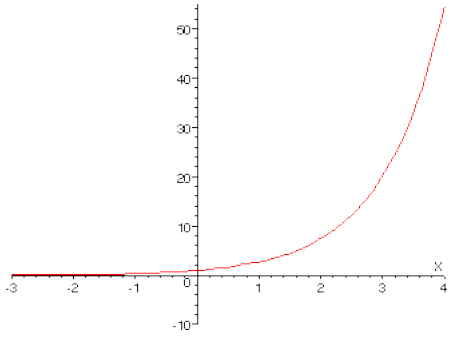
$$\log_b \frac{1}{c} = -\log_b c \quad \longleftrightarrow \quad b^{-m} = \frac{1}{b^m} \qquad \log_b a^r = r \log_b a \quad \longleftrightarrow \quad (b^m)^n = b^{mn}.$$

Graphs of Logarithmic and Exponential Functions



$f(x) = \ln x$

Notice that each curve is the reflection of the other about the line $y = x$.



$f(x) = e^x$

Limits of Logarithmic and Exponential Functions

1. $\lim_{x\to\infty} \dfrac{\ln x}{x} = 0$ ($\ln x$ grows more slowly than x).
2. $\lim_{x\to\infty} \dfrac{e^x}{x^n} = \infty$ for all positive integers n (e^x grows faster than x^n).
3. For $|x| \ll 1$, $\lim_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$.

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]